Published by
NIGERIAN SOCIETY OF PHYSICAL SCIENCES
Available online @ https://journal.nsps.org.ng/index.php/jnsps

J. Nig. Soc. Phys. Sci. (2026) 3662

Journal of the Nigerian Society of Physical Sciences

Factorization-driven operator framework for Δ_δ Gould–Hopper Appell polynomial families and their zeros via graphical representation

Shivtej Annaso Patil[a,b], Waseem Ahmad Khan[c], Wael Mahmoud Mohammad Salameh[d], Prakash Jadhav[e], Shahid Ahmad Wani[a,*]

[a] *Symbiosis Institute of Technology, Pune Campus, Symbiosis International (Deemed University), Pune, India*
[b] *General Engineering Department, DKTE Society's Textile and Engineering Institute, Ichalkaranji 416115, Maharashtra, India*
[c] *Department of Electrical Engineering, Prince Mohammad Bin Fahd University, P.O. Box 1664, Al Khobar 31952, Saudi Arabia*
[d] *Faculty of Information Technology, Abu Dhabi University, Abu Dhabi, United Arab Emirates*
[e] *Department of Mechanical Engineering, SRM University-AP, Amravati 522240, Andhra Pradesh, India*

Abstract

This article develops a factorization-based operator framework for two-variable discrete Gould–Hopper–Appell polynomials within Δ_δ-calculus. Following the Infeld–Hull factorization principle, the framework employs degree-dependent sequences of operators, rather than a single pair of ordinary quasi-monomial operators. From the generating function, we derive the recurrence relation, lowering and raising operators, and normalized integro-type operators. Their successive degree-matched compositions yield explicit difference, integro-difference, and partial-difference equations. Exact triangular Volterra integral representations are also established through the associated differential representation. The general results are specialized to Gould–Hopper–Bernoulli, Gould–Hopper–Euler, and Gould–Hopper–Genocchi families, with the Genocchi case treated separately. Numerical investigations of complex zeros, polynomial surfaces, generating-function reconstruction, and quantitative convergence as $\delta \to 0$ complement the analytical results and illustrate the discrete-to-continuous behavior of the proposed polynomial families.

DOI: 10.46481/jnsps.2026.3662

Keywords: Δ_δ Gould–Hopper Appell polynomials, Recurrence relation, Shift operators, Difference equations, Complex zeros

Article History:
Received: 02 July 2026
Received in revised form: 27 August 2026
Accepted for publication: 01 September 2026
Available online: 23 September 2026

Communicated by: P. Thakur

1. Introduction and preliminary framework

Special functions hold an indispensable position in mathematical physics and applied analysis. Polynomial sequences constructed via generating functions furnish unified tools for investigating orthogonality relations, approximation theory, and the construction of solutions to difference and differential equations. The Δ_δ-calculus provides a discrete counterpart of classical analysis: the forward difference operator $\Delta_\delta[f(u)] := f(u+\delta) - f(u)$ plays the role of the ordinary derivative, and letting $\delta \to 0$ recovers the standard continuous framework. The Δ_δ framework offers several decisive advantages over classical ($\delta = 1$) difference calculus: (i) the step size δ is a free parameter that can be tuned to match the mesh resolution of a given numerical scheme or the lattice constant of a physical model; (ii) the generating function $(1+\delta\tau)^{u/\delta}$ reduces to $e^{u\tau}$ as $\delta \to 0$,

*Corresponding Author Tel. No.: +91-700-653-2503.
Email address: `shahidwani177@gmail.com`; `shahid.wani@sitpune.edu.in` (Shahid Ahmad Wani)

providing a smooth deformation that links discrete and continuous theories and allows one to track how operators and equations deform as the grid is refined; and (iii) the richer algebraic structure of the Δ_δ-product rule, compared with the unit-step rule, yields operator identities with explicit δ-dependence that are invisible in the $\delta = 1$ theory.

These features make the Δ_δ framework particularly well suited for approximation theory, for numerical analysis of differential equations on non-uniform grids, and for the study of discrete dynamical systems whose step size is a modelling parameter rather than a fixed integer. Within this setting, Δ_δ-counterparts of Hermite, Gould–Hopper, Appell, and associated polynomial classes have drawn sustained research interest; see, for example, [1, 2, 3, 4, 5, 6, 7, 8, 9].

The factorization method, introduced systematically by Infeld and Hull [10], provides an operator-based procedure for treating eigenvalue and related differential-equation problems through the composition of suitable first-order operators. In the context of polynomial sequences, He and Ricci [11] adapted this approach to Appell polynomials by constructing lowering and raising operators whose composition yields the corresponding governing differential equation; subsequent extensions of this viewpoint may be found, for example, in [6].

Let $\{\mathcal{P}_n(u)\}_{n=0}^{\infty}$ be a polynomial sequence satisfying $\deg(\mathcal{P}_n(u)) = n$, where $n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$. Suppose that two operators Φ_n^- and Φ_n^+ act on this sequence according to

$$\Phi_n^-(\mathcal{P}_n(u)) = \mathcal{P}_{n-1}(u), \qquad \Phi_n^+(\mathcal{P}_n(u)) = \mathcal{P}_{n+1}(u). \tag{1}$$

The operators Φ_n^- and Φ_n^+ are referred to as the *lowering* (annihilation) and *raising* (creation) operators, respectively. This operator viewpoint is closely related to the poweroid/monomiality framework initiated by Steffensen [12] and subsequently developed for generalized polynomial families by Dattoli, Cesarano, and Sacchetti [13]. In this sense, a polynomial sequence endowed with compatible lowering and raising actions is described as *quasi-monomial*.

Applying the raising operator first and then the corresponding lowering operator gives the fundamental factorization relation

$$[\Phi_{n+1}^-\Phi_n^+]\mathcal{P}_n(u) = \mathcal{P}_n(u). \tag{2}$$

Indeed, Φ_n^+ maps $\mathcal{P}_n(u)$ to $\mathcal{P}_{n+1}(u)$, while Φ_{n+1}^- maps the resulting polynomial back to $\mathcal{P}_n(u)$. Hence, relation (2) is the factorized operator identity used to obtain the governing equation. For Appell-type polynomial systems, the use of such operator compositions to derive differential equations follows the approach of He and Ricci [11]; generalized finite-order versions were later investigated by Özarslan and Yilmaz Yaşar [6]. Once explicit realizations of Φ_n^- and Φ_n^+ are determined, their composition yields the corresponding governing equation. In the present Δ_δ setting, the analogous construction is implemented with difference operators rather than ordinary differential operators and will be used below to construct the relevant lowering and raising operators and, through their successive composition, to derive the governing equations for the polynomial family under consideration.

Figure 1 shows representative polynomial profiles, each normalized to unit maximum absolute value on the plotted range so that shapes of different degrees can be compared on one axis; it is used only for qualitative shape information and no claim that all n zeros are real is made. Table 6 and Figure 2 display the complete complex zero sets. Figure 3 compares Bernoulli and Euler patterns. Figure 4 gives the reproducible closed-form generating-function benchmark. Figure 5 shows the bivariate surfaces. Figure 6 reports a normed convergence error versus δ together with a fitted log–log rate. Figure 7 compares the three special families.

The classical Appell sequences $\{\mathcal{R}_n(v)\}$ are characterized by [14]

$$\mathcal{R}(\tau)\, e^{v\tau} = \sum_{n=0}^{\infty} \mathcal{R}_n(v)\, \frac{\tau^n}{n!}, \tag{3}$$

where $\mathcal{R}(\tau) = \sum_{n=0}^{\infty} \mathcal{R}_n\, \tau^n/n!$, $\mathcal{R}_0 \neq 0$, is analytic at $\tau = 0$. Such sequences satisfy $\mathcal{R}_n'(v) = n\,\mathcal{R}_{n-1}(v)$ and admit the binomial-type expansion $\mathcal{R}_n(v) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{R}_k\, v^{n-k}$. Three fundamental members—Bernoulli, Euler, and Genocchi polynomials—are collected in Table 1, with initial values of their number-sequences given in Table 2.

Almusawa [1] introduced the Δ_δ-bivariate Appell polynomials $\mathcal{A}_n^{(m)}(u, v; \delta)$ through the generating relation

$$\psi(\tau)\,(1+\delta\tau)^{u/\delta}\,(1+\delta\tau^m)^{v/\delta} = \sum_{n=0}^{\infty} \mathcal{A}_n^{(m)}(u, v; \delta)\,\frac{\tau^n}{n!}, \quad m \in \mathbb{N}_0, \tag{4}$$

where $\psi(\tau) = \sum_{k=0}^{\infty} \psi_k\, \tau^k/k!$, $\psi_0 \neq 0$, for the regular Appell branch. The Genocchi case, for which $\psi(0) = 0$, is treated separately by the shifted factorization $\psi(\tau) = \tau\widetilde{\psi}(\tau)$ with $\widetilde{\psi}(0) \neq 0$; see Section 4. Choosing $\psi(\tau) = 1$ in (4) produces the Δ_δ-Gould–Hopper polynomials $\mathcal{H}_n^{(m)}(u, v; \delta)$; see also [2, 4, 3]. The Δ_δ-forward difference operators act on $\mathcal{A}_n^{(m)}(u, v; \delta)$ according to

$${}_u\Delta_\delta \left\{\mathcal{A}_n^{(m)}(u, v; \delta)\right\} = n\delta\, \mathcal{A}_{n-1}^{(m)}(u, v; \delta), \tag{5}$$

$${}_v\Delta_\delta \left\{\mathcal{A}_n^{(m)}(u, v; \delta)\right\} = \frac{n!}{(n-m)!}\,\delta\, \mathcal{A}_{n-m}^{(m)}(u, v; \delta), \tag{6}$$

For later factorization steps we use right inverses only on the polynomial sequence space $\mathscr{P} = \operatorname{span}\{\mathcal{A}_n^{(m)} : n \geq 0\}$, with the normalization that the inverse has zero component in the null space of the corresponding forward difference. Thus

$${}_u\Delta_\delta^{-1} \left\{\mathcal{A}_{n-1}^{(m)}(u, v; \delta)\right\} = \frac{1}{n\delta}\, \mathcal{A}_n^{(m)}(u, v; \delta), \tag{7}$$

$${}_v\Delta_\delta^{-1} \left\{\mathcal{A}_{n-m}^{(m)}(u, v; \delta)\right\} = \frac{(n-m)!}{n!\,\delta}\, \mathcal{A}_n^{(m)}(u, v; \delta), \qquad n \geq m. \tag{8}$$

Equivalently, on this normalized range, ${}_u\Delta_\delta({}_u\Delta_\delta^{-1} f) = f$ and ${}_u\Delta_\delta^{-1}({}_u\Delta_\delta f) = f - \Pi_u f$, where Π_u is the projection onto the u-independent null space. Every cancellation below is made only after this normalization has been specified. For comparison with ordinary quasi-monomiality, the generating-function symbols often written for this family are [1]

$$\hat{M}(\tau) = \frac{u}{1+\delta\tau} + \frac{m\,v\,\tau^{m-1}}{1+\delta\tau^m} + \frac{\psi'(\tau)}{\psi(\tau)}, \qquad \hat{D}(\tau) = \frac{\log(1+\delta\tau)}{n\delta}. \tag{9}$$

Table 1: Principal members of the Appell family [15, 16].

Name	$\mathcal{R}(\tau)$	Generating function	Numbers
Bernoulli $B_n(v)$	$\frac{\tau}{e^\tau - 1}$	$\frac{\tau}{e^\tau - 1} e^{v\tau} = \sum_{n=0}^{\infty} B_n(v) \frac{\tau^n}{n!}$	$B_n := B_n(0)$
Euler $E_n(v)$	$\frac{2}{e^\tau + 1}$	$\frac{2}{e^\tau + 1} e^{v\tau} = \sum_{n=0}^{\infty} E_n(v) \frac{\tau^n}{n!}$	$E_n := 2^n E_n\left(\frac{1}{2}\right)$
Genocchi $G_n(v)$	$\frac{2\tau}{e^\tau + 1}$	$\frac{2\tau}{e^\tau + 1} e^{v\tau} = \sum_{n=0}^{\infty} G_n(v) \frac{\tau^n}{n!}$	$G_n := G_n(0)$

Table 2: Initial values of Bernoulli, Euler and Genocchi numbers [17, 16].

n	0	1	2	3	4
B_n	1	$\pm\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$
E_n	1	0	-1	0	5
G_n	0	1	-1	0	1

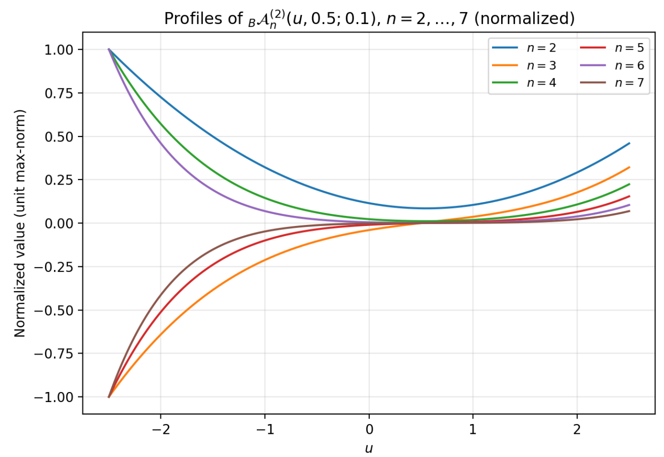

Figure 1: Profiles of $_B\mathcal{A}_n^{(2)}(u, 0.5; 0.1)$ for $n = 2, \ldots, 7$, each normalized to unit maximum absolute value on $u \in [-2.5, 2.5]$; the curves illustrate shape changes only.

These are *generating symbols*, not the operators used in our proofs: they depend on τ (and $\hat{D}$ also on n), and therefore do not by themselves define an operator on the polynomial variable. The present work instead follows the Infeld–Hull–He–Ricci factorization philosophy: for each degree n we construct a *sequence* of normalized operators $\{L_n, R_n\}$ acting on $\mathscr{P}_n$, verify $L_n\mathcal{A}_n = \mathcal{A}_{n-1}$ and $R_n\mathcal{A}_n = \mathcal{A}_{n+1}$, and only then form the degree-matched composition

$$(L_{n+1} \circ R_n)\mathcal{A}_n := L_{n+1}(R_n(\mathcal{A}_n)) = \mathcal{A}_n. \tag{10}$$

Thus the factorization here is deliberately more structured than the ordinary two-operator quasi-monomial identity: the normalization changes with the degree and several compatible u-, v-, and integro-factorizations coexist.

The practical relevance of the polynomial families studied here is threefold. First, the recurrence relations and shift operators derived for the Δ_δ-Gould–Hopper Appell family provide a systematic basis for constructing stable quadrature rules and interpolation schemes on non-uniform grids, which is a direct application in approximation theory. Second, the governing difference equations can model discrete dynamical systems in which the step size δ is a tunable parameter reflecting, for example, the lattice constant of a crystal or the sampling interval of a time series; the zero distributions studied numerically in Section 5 then give direct information about the spectral properties of such systems. Third, the closed-form Volterra integral equations enable the design of spectral methods for discrete boundary value problems, since they recast the polynomial evaluation problem into a fixed-point iteration accessible by standard numerical analysis techniques.

The paper proceeds as follows. Section 2 derives the recurrence relation (Theorem 2.1) and then presents, each as a distinct theorem with its own proof, the u-lowering operator, the v-lowering operator, the u-raising operator, and the v-raising operator, together with the governing difference equation obtained from their factorized composition. Section 3 establishes the two integro-lowering operators, the two integro-raising operators, the integro-difference equation, and the partial difference equation via the integro-operators, each formulated as a separate theorem. Section 4 records all results for the three special cases in comprehensive tables containing every equation written out explicitly, including the partial difference equations. Section 5 derives the Volterra integral equation and provides numerical illustrations, including a reproducible root table.

2. Main results

For compactness write $D_u := {}_u\Delta_\delta/\delta$ and $D_v := {}_v\Delta_\delta/\delta$. Then $D_u\mathcal{A}_n^{(m)} = n\mathcal{A}_{n-1}^{(m)}$ and $D_v\mathcal{A}_n^{(m)} = n!/(n-m)!\,\mathcal{A}_{n-m}^{(m)}$. These are exact finite-difference identities; no ordinary derivative is identified with a forward difference.

Theorem 2.1 (Recurrence relation). *Let $\delta > 0$ and $m \in \mathbb{N}$ be fixed, let $u, v \in \mathbb{R}$, and let $\psi(\tau) = \sum_{k\geq 0} \psi_k \tau^k / k!$ be analytic on a disk $|\tau| < R_\psi$ with $\psi_0 \neq 0$ (the regular Appell branch).*

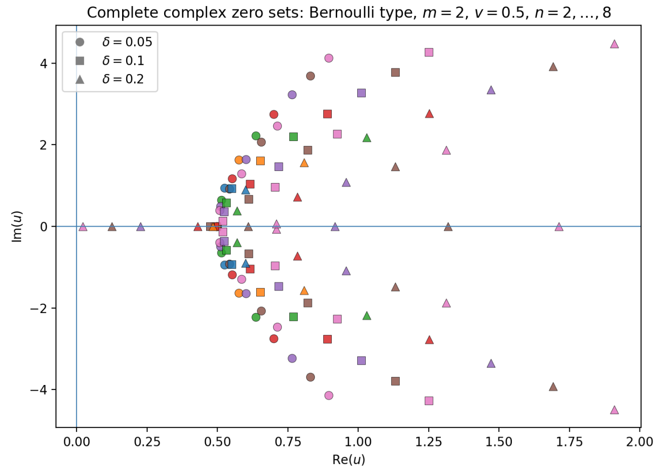

Figure 2: Complete complex zeros (real and non-real) of ${}_B\mathcal{A}_n^{(2)}(u, 0.5; \delta)$ for $n = 2, \ldots, 8$ and $\delta \in \{0.05, 0.10, 0.20\}$, tabulated in Table 6. Roots are obtained from the full coefficient vector; conjugate pairs are retained and residuals are below 10^{-8}.

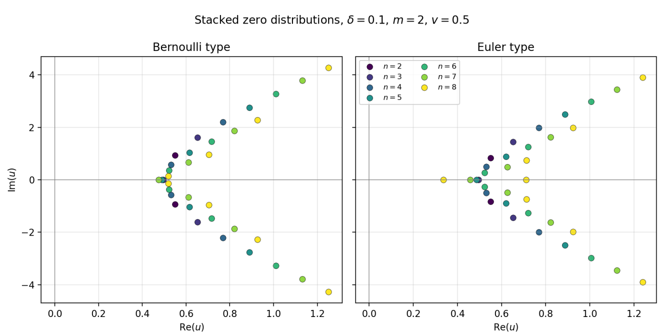

Figure 3: Stacked zero distributions for Bernoulli-type (left) and Euler-type (right), $\delta = 0.1$, $m = 2$, $v = 0.5$.

Then, on the disk $|\tau| < R := \min(R_\psi, 1/(\delta), \delta^{-1/m})$, the map $\tau \mapsto \psi(\tau)(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta}$ is single-valued and analytic (the two binomial factors are defined by their principal branch, which is analytic for $|\delta\tau| < 1$ and $|\delta\tau^m| < 1$ respectively), so the generating relation (4) defines $\mathcal{A}_n^{(m)}(u, v; \delta) \in \mathbb{R}$ unambiguously for every $n \geq 0$ as $n!$ times the n-th Taylor coefficient at $\tau = 0$; in particular the term-by-term differentiation used below is justified on $|\tau| < R$. Under these hypotheses, $\mathcal{A}_n^{(m)}(u, v; \delta)$ satisfies, for every $n \geq 0$, the exact recurrence

$$\begin{aligned}\mathcal{A}_{n+1}^{(m)}(u, v; \delta) = &\sum_{k=0}^{n} \binom{n}{k} \lambda_{n-k,\delta}\, \mathcal{A}_k^{(m)}(u, v; \delta) \\ &+ u\, n! \sum_{j=0}^{n} \frac{(-\delta)^j}{(n-j)!}\, \mathcal{A}_{n-j}^{(m)}(u, v; \delta) \\ &+ mv\, n! \sum_{j=0}^{\lfloor (n-m+1)/m \rfloor} \frac{(-\delta)^j}{(n-mj-m+1)!} \\ &\quad \times \mathcal{A}_{n-mj-m+1}^{(m)}(u, v; \delta).\end{aligned} \tag{11}$$

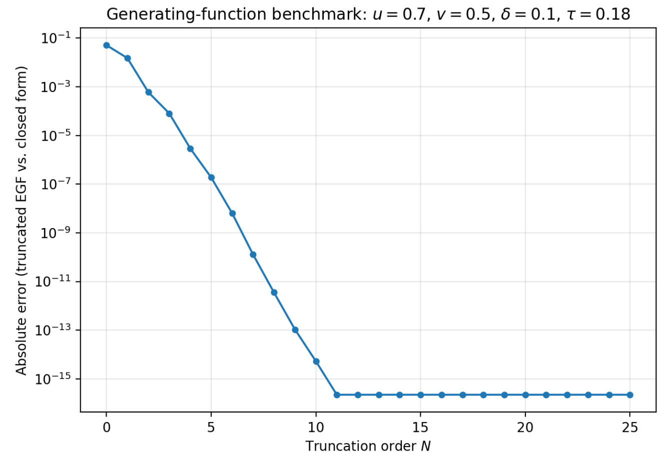

Figure 4: Generating-function benchmark at $u = 0.7$, $v = 0.5$, $\delta = 0.1$, $\tau = 0.18$: truncated EGF versus the closed-form left side of (4). Calculations use IEEE double precision; the plotted error is absolute error.

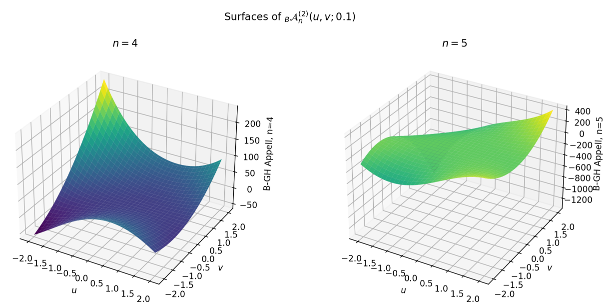

Figure 5: Three-dimensional surface plots of ${}_B\mathcal{A}_4^{(2)}(u, v; 0.1)$ (left) and ${}_B\mathcal{A}_5^{(2)}(u, v; 0.1)$ (right).

where $\lambda_{k,\delta}$ is defined by (12), and where the last sum is understood to be empty — hence equal to 0 — whenever $\lfloor (n-m+1)/m \rfloor < 0$, i.e. whenever $n < m-1$. (No claim is made that ψ'/ψ or the recurrence extends to the Genocchi case $\psi_0 = 0$; that case is treated separately in Section 4 through the shifted regular function $\widetilde{\psi} = \psi/\tau$, to which this theorem applies verbatim.)

$$\frac{\psi'(\tau)}{\psi(\tau)} = \sum_{k=0}^{\infty} \lambda_{k,\delta} \frac{\tau^k}{k!}, \qquad |\tau| < R, \tag{12}$$

the right-hand side being the Taylor expansion of the (analytic, since $\psi_0 \neq 0$) logarithmic derivative of ψ.

Proof. Both sides of (4) are analytic functions of τ on $|\tau| < R$ by the hypotheses above, so they may be differentiated term by term on this disk. Differentiating the generating function (4)

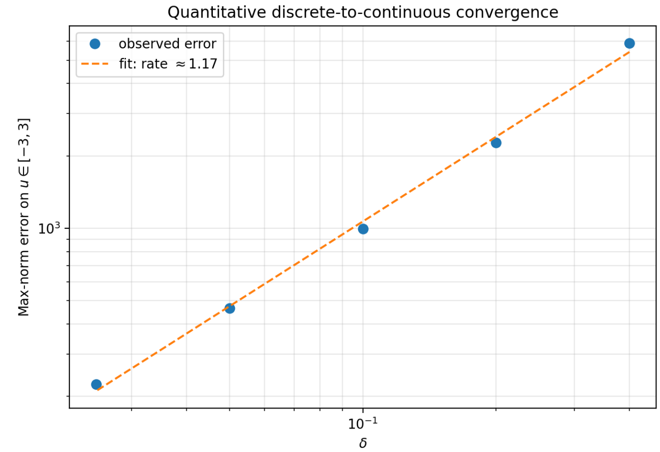

Figure 6: Quantitative discrete-to-continuous convergence for ${}_{B}\mathcal{A}_6^{(2)}$ with $v = 0.5$: maximum-norm error on $u \in [-3, 3]$, relative to the $\delta \to 0$ limit ${}_{B}\mathcal{H}_6^{(2)}(u, v) = \lim_{\delta\to 0} {}_{B}\mathcal{A}_6^{(2)}(u, v; \delta)$ (classical two-variable Gould–Hopper–Bernoulli polynomial), versus $\delta \in \{0.4, 0.2, 0.1, 0.05, 0.025\}$. A log–log least-squares fit gives an observed rate of approximately 1.17 over the displayed mesh sizes, consistent with the first-order term $-\frac{1}{2}\delta\tau^2$ in $\delta^{-1}\log(1 + \delta\tau) - \tau = O(\delta)$.

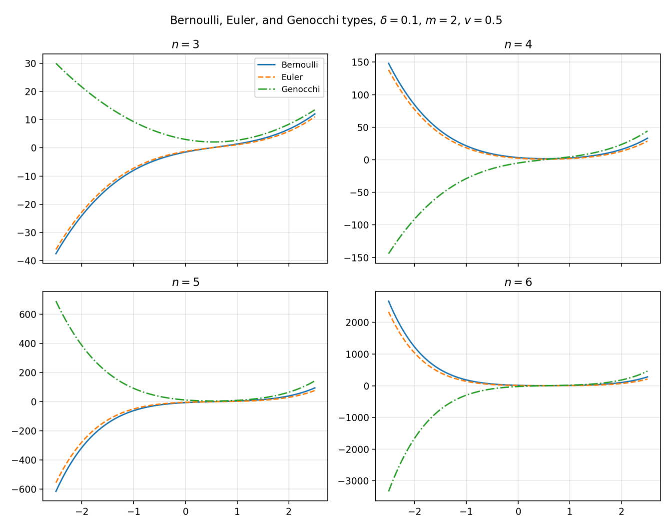

Figure 7: Bernoulli, Euler, and Genocchi types for $n = 3, 4, 5, 6$, $\delta = 0.1$, $m = 2$, $v = 0.5$.

with respect to τ, we obtain

$$\begin{aligned}
&\frac{\psi'(\tau)}{\psi(\tau)}\psi(\tau)(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta}\\
&+\frac{u}{1+\delta\tau}\psi(\tau)(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta}\\
&+\frac{mv\tau^{m-1}}{1+\delta\tau^m}\psi(\tau)(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta}\\
&=\sum_{n=0}^{\infty}\mathcal{A}_{n+1}^{(m)}(u,v;\delta)\frac{\tau^n}{n!}.
\end{aligned}\tag{13}$$

Using (4) and (12), equation (13) can be written as

$$\begin{aligned}
&\left[\sum_{k=0}^{\infty}\lambda_{k,\delta}\frac{\tau^k}{k!}+\frac{u}{1+\delta\tau}+\frac{mv\tau^{m-1}}{1+\delta\tau^m}\right]\\
&\times\sum_{r=0}^{\infty}\mathcal{A}_r^{(m)}(u,v;\delta)\frac{\tau^r}{r!}\\
&=\sum_{n=0}^{\infty}\mathcal{A}_{n+1}^{(m)}(u,v;\delta)\frac{\tau^n}{n!}.
\end{aligned}\tag{14}$$

For the first term on the left-hand side, the Cauchy product gives

$$\left(\sum_{k=0}^{\infty}\lambda_{k,\delta}\frac{\tau^k}{k!}\right)\left(\sum_{r=0}^{\infty}\mathcal{A}_r^{(m)}\frac{\tau^r}{r!}\right)=\sum_{n=0}^{\infty}\left[\sum_{k=0}^{n}\binom{n}{k}\lambda_{n-k,\delta}\mathcal{A}_k^{(m)}\right]\frac{\tau^n}{n!}. \tag{15}$$

Further, using

$$\frac{1}{1+\delta\tau}=\sum_{j=0}^{\infty}(-\delta)^j\tau^j,\qquad \frac{1}{1+\delta\tau^m}=\sum_{j=0}^{\infty}(-\delta)^j\tau^{mj}, \tag{16}$$

we obtain

$$\frac{u}{1+\delta\tau}\sum_{r=0}^{\infty}\mathcal{A}_r^{(m)}\frac{\tau^r}{r!}=\sum_{n=0}^{\infty}\left[u\,n!\sum_{j=0}^{n}\frac{(-\delta)^j}{(n-j)!}\mathcal{A}_{n-j}^{(m)}\right]\frac{\tau^n}{n!}. \tag{17}$$

Similarly,

$$\begin{aligned}
\frac{mv\tau^{m-1}}{1+\delta\tau^m}\sum_{r=0}^{\infty}\mathcal{A}_r^{(m)}\frac{\tau^r}{r!}&=mv\sum_{j=0}^{\infty}(-\delta)^j\tau^{mj+m-1}\sum_{r=0}^{\infty}\mathcal{A}_r^{(m)}\frac{\tau^r}{r!}\\
&=\sum_{n=0}^{\infty}\left[mvn!\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}\frac{(-\delta)^j}{(n-mj-m+1)!}\right.\\
&\qquad\left.\times\mathcal{A}_{n-mj-m+1}^{(m)}\right]\frac{\tau^n}{n!}.
\end{aligned}\tag{18}$$

Here the upper limit follows from the requirement

$$n-mj-m+1\geq 0,$$

and hence

$$j\leq\frac{n-m+1}{m}.$$

If no non-negative integer satisfies this condition, the corresponding sum is understood to be empty.

Substituting (15)–(18) into (14) and comparing the coefficients of $\tau^n/n!$ on both sides gives

$$\begin{aligned}
\mathcal{A}_{n+1}^{(m)}(u,v;\delta)&=\sum_{k=0}^{n}\binom{n}{k}\lambda_{n-k,\delta}\mathcal{A}_k^{(m)}(u,v;\delta)\\
&\quad+u\,n!\sum_{j=0}^{n}\frac{(-\delta)^j}{(n-j)!}\mathcal{A}_{n-j}^{(m)}(u,v;\delta)\\
&\quad+mv\,n!\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}\frac{(-\delta)^j}{(n-mj-m+1)!}\\
&\qquad\times\mathcal{A}_{n-mj-m+1}^{(m)}(u,v;\delta).
\end{aligned}$$

which is precisely (11). □

Theorem 2.2. *The u-lowering member of the factorization sequence is*

$$_uM_n^- := \frac{1}{n\delta}\,{}_u\Delta_\delta, \qquad n \geq 1, \tag{19}$$

and ${}_uM_n^-\{\mathcal{A}_n^{(m)}\} = \mathcal{A}_{n-1}^{(m)}$.

Proof. Applying ${}_u\Delta_\delta$ to (4) and using

$${}_u\Delta_\delta(1+\delta\tau)^{u/\delta} = \delta\tau(1+\delta\tau)^{u/\delta},$$

we obtain

$$\sum_{n\geq 0} {}_u\Delta_\delta \mathcal{A}_n^{(m)} \frac{\tau^n}{n!} = \delta\tau \sum_{n\geq 0} \mathcal{A}_n^{(m)} \frac{\tau^n}{n!}.$$

Comparison of coefficients gives

$${}_u\Delta_\delta \mathcal{A}_n^{(m)} = n\delta \mathcal{A}_{n-1}^{(m)}.$$

Division by $n\delta$ proves (19). This is an exact forward-difference identity. □

Define on the normalized sequence space $\mathscr{P}$ the composite operator

$$Q_m := \delta^{m-2}\,{}_v\Delta_\delta\,{}_u\Delta_\delta^{-(m-1)}. \tag{20}$$

The right inverse in (20) is the normalized inverse defined after (8); hence no arbitrary summation constant is present.

Theorem 2.3. *The v-based lowering member of the factorization sequence is*

$${}_vM_n^- := \frac{1}{n}Q_m = \frac{\delta^{m-2}}{n}\,{}_v\Delta_\delta\,{}_u\Delta_\delta^{-(m-1)}, \qquad n \geq 1, \tag{21}$$

and ${}_vM_n^-\{\mathcal{A}_n^{(m)}\} = \mathcal{A}_{n-1}^{(m)}$.

Proof. For every $n \geq 1$, repeated use of the normalized right inverse in (7) gives

$${}_u\Delta_\delta^{-(m-1)} \mathcal{A}_n^{(m)} = \frac{n!}{(n+m-1)!\,\delta^{m-1}} \mathcal{A}_{n+m-1}^{(m)}.$$

Applying the exact v-difference identity to the resulting degree $n+m-1$ polynomial yields

$$\begin{aligned}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)} \mathcal{A}_n^{(m)} &= \frac{n!}{(n+m-1)!\delta^{m-1}} \frac{(n+m-1)!}{(n-1)!} \\ &\quad \times \delta\,\mathcal{A}_{n-1}^{(m)} = n\delta^{2-m}\mathcal{A}_{n-1}^{(m)}.\end{aligned}$$

Multiplication by δ^{m-2}/n proves (21). This argument is valid for all $n \geq 1$ and does not require a negative factorial or an unrestricted commutation through the inverse difference. □

Theorem 2.4. *The u-raising member is*

$$\begin{aligned}{}_uM_n^+ := &\sum_{k=0}^{n} \frac{\lambda_{n-k,\delta}}{(n-k)!\delta^{n-k}}\,{}_u\Delta_\delta^{n-k} \\ &+ u\sum_{j=0}^{n} (-1)^j\,{}_u\Delta_\delta^j \\ &+ mv \sum_{j=0}^{\lfloor (n-m+1)/m \rfloor} \frac{(-1)^j}{\delta^{(m-1)(j+1)}}\,{}_u\Delta_\delta^{m(j+1)-1}.\end{aligned} \tag{22}$$

and ${}_uM_n^+\{\mathcal{A}_n^{(m)}\} = \mathcal{A}_{n+1}^{(m)}$.

Proof. Repeated use of (19) gives

$${}_u\Delta_\delta^r \mathcal{A}_n^{(m)} = \frac{n!}{(n-r)!}\delta^r \mathcal{A}_{n-r}^{(m)}, \qquad 0 \leq r \leq n,$$

and therefore

$$\mathcal{A}_k^{(m)} = \frac{k!}{n!\delta^{n-k}}\,{}_u\Delta_\delta^{n-k} \mathcal{A}_n^{(m)}.$$

Substituting these expressions for all lower-index polynomials in (11) and collecting the operators acting on $\mathcal{A}_n^{(m)}$ gives (22). Hence

$${}_uM_n^+\{\mathcal{A}_n^{(m)}\} = \mathcal{A}_{n+1}^{(m)}.$$

□

Theorem 2.5. *The companion v-raising member is*

$$\begin{aligned}{}_vM_n^+ := &\sum_{k=0}^{n} \frac{\lambda_{n-k,\delta}}{(n-k)!} \left(\delta^{m-2}\,{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{n-k} \\ &+ u\sum_{j=0}^{n} (-\delta)^j \left(\delta^{m-2}\,{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^j \\ &+ mv \sum_{j=0}^{\lfloor (n-m+1)/m \rfloor} (-\delta)^j \left(\delta^{m-2}\,{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{mj+m-1}.\end{aligned} \tag{23}$$

where the products are interpreted through the degree-matched sequence action; equivalently one may substitute $\mathcal{A}_k^{(m)} = k!\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{n-k} \mathcal{A}_n^{(m)}/n!$ *directly in (11). It satisfies* ${}_vM_n^+\{\mathcal{A}_n^{(m)}\} = \mathcal{A}_{n+1}^{(m)}$.

Proof. By Theorem 2.3,

$$Q_m \mathcal{A}_r^{(m)} = r\mathcal{A}_{r-1}^{(m)},$$

so that

$$Q_m^k \mathcal{A}_n^{(m)} = \frac{n!}{(n-k)!}\mathcal{A}_{n-k}^{(m)}.$$

Replacing each lower-index polynomial in (11) by this formula gives the expression (23). Therefore

$${}_vM_n^+\{\mathcal{A}_n^{(m)}\} = \mathcal{A}_{n+1}^{(m)}.$$

□

Theorem 2.6. *The exact governing difference equation satisfied by* $\mathcal{A}_n^{(m)}(u,v;\delta)$ *is*

$$\begin{aligned}&\left[\frac{1}{(n+1)\delta}\,{}_u\Delta_\delta\left\{\sum_{k=0}^{n} \frac{\lambda_{n-k,\delta}}{(n-k)!\delta^{n-k}}\,{}_u\Delta_\delta^{n-k} + u\sum_{j=0}^{n}(-1)^j\,{}_u\Delta_\delta^j\right.\right. \\ &\left.\left. + mv \sum_{j=0}^{\lfloor (n-m+1)/m \rfloor} \frac{(-1)^j}{\delta^{(m-1)(j+1)}}\,{}_u\Delta_\delta^{m(j+1)-1}\right\} - I\right]\mathcal{A}_n^{(m)}(u,v;\delta) = 0.\end{aligned} \tag{24}$$

Equivalently, use of the forward-difference product rule expands this expression into a finite linear combination of forward differences of $\mathcal{A}_n^{(m)}$ *only.*

Proof. The adjacent raising and lowering relations give

$$_uM_n^+\mathcal{A}_n^{(m)} = \mathcal{A}_{n+1}^{(m)}, \qquad _uM_{n+1}^-\mathcal{A}_{n+1}^{(m)} = \mathcal{A}_n^{(m)}.$$

Thus

$$(_uM_{n+1}^- \circ {_uM_n^+})\mathcal{A}_n^{(m)} = \mathcal{A}_n^{(m)}.$$

Substituting (19) and (22), and using

$$_u\Delta_\delta[fg] = f(u+\delta)_u\Delta_\delta g + g\, _u\Delta_\delta f,$$

then collecting equal powers of $_u\Delta_\delta$, yields (24). □

Remark 2.1 (Verification at low degrees)**.** *From the defining generating function,*

$$\mathcal{A}_0^{(m)}(u,v;\delta) = \psi_0,$$

and the exact u-difference identity gives

$$_u\Delta_\delta\, \mathcal{A}_1^{(m)} = \delta\, \mathcal{A}_0^{(m)}, \qquad _u\Delta_\delta\, \mathcal{A}_2^{(m)} = 2\delta\, \mathcal{A}_1^{(m)}.$$

Hence the normalized lowering operator satisfies

$$_uM_1^-\mathcal{A}_1^{(m)} = \mathcal{A}_0^{(m)}, \qquad _uM_2^-\mathcal{A}_2^{(m)} = \mathcal{A}_1^{(m)}.$$

Using the explicit recurrence defining the raising operator, one similarly checks

$$_uM_0^+\mathcal{A}_0^{(m)} = \mathcal{A}_1^{(m)}, \qquad _uM_1^+\mathcal{A}_1^{(m)} = \mathcal{A}_2^{(m)}.$$

Consequently,

$$(_uM_1^- \circ {_uM_0^+})\mathcal{A}_0^{(m)} = \mathcal{A}_0^{(m)}, \qquad (_uM_2^- \circ {_uM_1^+})\mathcal{A}_1^{(m)} = \mathcal{A}_1^{(m)}.$$

These direct checks confirm the degree-matched factorization and the governing equation at the first admissible degrees, making the resolution of the normalization question fully transparent.

3. Integro-operators and associated equations

The inverse differences in this section are the normalized right inverses on $\mathscr{P}$ defined after (8). Consequently, all compositions are identities on the stated sequence space; no unrestricted cancellation of an inverse difference is used.

Theorem 3.1. *The u-integro-lowering operator is*

$$_u\mathcal{M}_n^- := \frac{1}{n}Q_m = \frac{\delta^{m-2}}{n}\, _v\Delta_{\delta u}\Delta_\delta^{-(m-1)}, \tag{25}$$

and $_u\mathcal{M}_n^-\mathcal{A}_n^{(m)} = \mathcal{A}_{n-1}^{(m)}$.

Proof. From the exact relation used in Theorem 2.3,

$$_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\mathcal{A}_n^{(m)} = n\delta^{2-m}\mathcal{A}_{n-1}^{(m)}.$$

Multiplying by δ^{m-2}/n gives

$$_u\mathcal{M}_n^-\mathcal{A}_n^{(m)} = \mathcal{A}_{n-1}^{(m)},$$

which proves (25). □

Theorem 3.2. *A second, genuinely v-integral lowering factorization is*

$$_v\mathcal{M}_n^- := \frac{1}{n\delta^m}\, _v\Delta_\delta^{-1}\, _u\Delta_\delta^{m+1}, \tag{26}$$

and $_v\mathcal{M}_n^-\mathcal{A}_n^{(m)} = \mathcal{A}_{n-1}^{(m)}$ *whenever* $n \geq m+1$*; the finitely many lower degrees are defined directly by* (19)*.*

Proof. Repeated u-lowering gives

$$_u\Delta_\delta^{m+1}\mathcal{A}_n^{(m)} = \frac{n!}{(n-m-1)!}\delta^{m+1}\mathcal{A}_{n-m-1}^{(m)}.$$

Using the normalized inverse v-difference,

$$_v\Delta_\delta^{-1}\mathcal{A}_{n-m-1}^{(m)} = \frac{(n-m-1)!}{(n-1)!\delta}\mathcal{A}_{n-1}^{(m)}.$$

Consequently,

$$_v\Delta_\delta^{-1}\, _u\Delta_\delta^{m+1}\mathcal{A}_n^{(m)} = n\delta^m\mathcal{A}_{n-1}^{(m)}.$$

Division by $n\delta^m$ proves (26). □

Theorem 3.3. *The u-integro-raising operator is obtained from the recurrence by the exact substitutions generated by* (25)*:*

$$\begin{aligned} _u\mathcal{M}_n^+ := {} & \sum_{k=0}^{n}\frac{\lambda_{n-k,\delta}}{(n-k)!}\left(\delta^{m-2}\, _v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{n-k} \\ & + u\sum_{j=0}^{n}(-\delta)^j\left(\delta^{m-2}\, _v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^j \\ & + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}(-\delta)^j\left(\delta^{m-2}\, _v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{mj+m-1}. \end{aligned} \tag{27}$$

Then $_u\mathcal{M}_n^+\mathcal{A}_n^{(m)} = \mathcal{A}_{n+1}^{(m)}$.

Proof. Iteration of Theorem 3.1 gives

$$\left(\delta^{m-2}\, _v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^r\mathcal{A}_n^{(m)} = \frac{n!}{(n-r)!}\mathcal{A}_{n-r}^{(m)}.$$

Substituting this relation for every lower-index polynomial in (11) and simplifying the factorials gives (27). Hence

$$_u\mathcal{M}_n^+\mathcal{A}_n^{(m)} = \mathcal{A}_{n+1}^{(m)}.$$

□

Theorem 3.4. *The v-integro-raising operator for* $\mathcal{A}_n^{(m)}(u,v;\delta)$ *is*

$$\begin{aligned} _v\mathcal{M}_n^+ := {} & \sum_{k=0}^{n}\frac{\lambda_{n-k,\delta}}{(n-k)!\,\delta^{m(n-k)}}\left(_v\Delta_\delta^{-1}\, _u\Delta_\delta^{m+1}\right)^{n-k} \\ & + u\sum_{j=0}^{n}\frac{(-\delta)^j}{\delta^{mj}}\left(_v\Delta_\delta^{-1}\, _u\Delta_\delta^{m+1}\right)^j \\ & + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}\frac{(-\delta)^j}{\delta^{m(mj+m-1)}}\left(_v\Delta_\delta^{-1}\, _u\Delta_\delta^{m+1}\right)^{mj+m-1}. \end{aligned} \tag{28}$$

and it satisfies

$$_v\mathcal{M}_n^+\mathcal{A}_n^{(m)}(u,v;\delta) = \mathcal{A}_{n+1}^{(m)}(u,v;\delta).$$

Here every inverse v-difference is the normalized right inverse defined on $\mathscr{P}$.

Proof. From Theorem 3.2, repeated normalized lowering yields

$$\left({}_v\Delta_\delta^{-1}\,{}_u\Delta_\delta^{m+1}\right)^r \mathcal{A}_n^{(m)} = \frac{n!}{(n-r)!}\delta^{mr}\mathcal{A}_{n-r}^{(m)}.$$

Thus

$$\mathcal{A}_{n-r}^{(m)} = \frac{(n-r)!}{n!\delta^{mr}}\left({}_v\Delta_\delta^{-1}\,{}_u\Delta_\delta^{m+1}\right)^r \mathcal{A}_n^{(m)}.$$

Substitution in (11) gives the explicit expression (28), and hence

$${}_v\mathcal{M}_n^{+}\mathcal{A}_n^{(m)} = \mathcal{A}_{n+1}^{(m)}.$$

□

Theorem 3.5. *The exact integro-difference equation satisfied by $\mathcal{A}_n^{(m)}(u,v;\delta)$ is*

$$\begin{aligned}\Bigg[&\frac{Q_m}{n+1}\Bigg\{\sum_{k=0}^{n}\frac{\lambda_{n-k,\delta}}{(n-k)!}\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{n-k}\\ &+u\sum_{j=0}^{n}(-\delta)^j\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^j\\ &+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}(-\delta)^j\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{mj+m-1}\Bigg\}-I\Bigg]\\ &\times\mathcal{A}_n^{(m)}(u,v;\delta)=0.\end{aligned}\tag{29}$$

All inverse differences in this expression are the normalized right inverses on $\mathscr{P}$ fixed in the preceding definitions.

Proof. The adjacent integro-raising and integro-lowering actions satisfy

$${}_u\mathcal{M}_n^{+}\mathcal{A}_n^{(m)} = \mathcal{A}_{n+1}^{(m)}, \qquad {}_u\mathcal{M}_{n+1}^{-}\mathcal{A}_{n+1}^{(m)} = \mathcal{A}_n^{(m)}.$$

Substituting the explicit expressions (25) and (27) into this composition and moving the right-hand side to the left gives exactly (29). The inverse differences are the normalized right inverses fixed on $\mathscr{P}$. □

Theorem 3.6. *Applying ${}_u\Delta_\delta^{n+m-1}$ to the explicit integro-difference relation gives the partial difference equation*

$$\begin{aligned}{}_u\Delta_\delta^{n+m-1}\Bigg[&\frac{Q_m}{n+1}\Bigg\{\sum_{k=0}^{n}\frac{\lambda_{n-k,\delta}}{(n-k)!}\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{n-k}\\ &+u\sum_{j=0}^{n}(-\delta)^j\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^j\\ &+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}(-\delta)^j\left(\delta^{m-2}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}\right)^{mj+m-1}\Bigg\}-I\Bigg]\\ &\times\mathcal{A}_n^{(m)}(u,v;\delta)=0.\end{aligned}\tag{30}$$

After the normalized cancellations described in the proof, this relation contains no uncontrolled inverse-difference term.

Proof. Apply ${}_u\Delta_\delta^{n+m-1}$ to (29). The normalized inverse powers of ${}_u\Delta_\delta$ cancel within their prescribed sequence action, leaving only forward differences in u and v. Collecting the remaining terms gives (30). □

4. Special cases and tabulated results

We now specialize (4) to the Bernoulli and Euler regular Appell branches and treat the Genocchi branch by the shifted regularization described below. For compactness define

$$\beta_r := -\frac{B_{r+1}(1)}{r+1}, \qquad \varepsilon_r := \frac{\mathcal{E}_r}{2}, \qquad r \geq 0,$$

so that $\beta_0 = \varepsilon_0 = -\frac{1}{2}$, and

$$\frac{d}{d\tau}\log\!\left(\frac{\tau}{e^\tau-1}\right) = \sum_{r\geq 0}\beta_r\frac{\tau^r}{r!}, \qquad \frac{d}{d\tau}\log\!\left(\frac{2}{e^\tau+1}\right) = \sum_{r\geq 0}\varepsilon_r\frac{\tau^r}{r!}.$$

Here

$$\mathcal{E}_r = -\frac{1}{2^r}\sum_{j=0}^{r}\binom{r}{j}E_{r-j}.$$

Throughout the tables below we use the already defined degree-lowering operator

$$Q_m := \delta^{m-2}\,{}_v\Delta_\delta\,{}_u\Delta_\delta^{-(m-1)}, \qquad Q_m\mathcal{A}_r^{(m)} = r\mathcal{A}_{r-1}^{(m)}.$$

The specialized formulas are kept in the same summation structure as Theorems 2.1–3.6; in particular, no $j = 0$ term from the v-sum is replaced by an expression proportional to $\mathcal{A}_n^{(m)}$.

Genocchi shifted regularization. For

$$\frac{2\tau}{e^\tau+1}(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta} = \sum_{n\geq 0}{}_{Gc}\mathcal{A}_n^{(m)}\frac{\tau^n}{n!},$$

write

$$\widetilde{\psi}(\tau) = \frac{2}{e^\tau+1}, \qquad \widetilde{\psi}(\tau)(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta} = \sum_{r\geq 0}\widetilde{\mathcal{A}}_r^{(m)}\frac{\tau^r}{r!}.$$

Then

$${}_{Gc}\mathcal{A}_0^{(m)} = 0, \qquad {}_{Gc}\mathcal{A}_n^{(m)} = n\widetilde{\mathcal{A}}_{n-1}^{(m)}, \quad n \geq 1. \tag{31}$$

No logarithmic derivative of $2\tau/(e^\tau+1)$ is used.

5. Exact Volterra integral representations and numerical illustrations

The full root set used to produce Figure 2 is recorded in Table 6 for reproducibility.

The ordinary derivative of the discrete generating factor is $\partial_u(1+\delta\tau)^{u/\delta} = \delta^{-1}\log(1+\delta\tau)(1+\delta\tau)^{u/\delta}$; therefore it must not be replaced by $\tau(1+\delta\tau)^{u/\delta}$ when $\delta \neq 0$. Expanding the logarithm gives the following exact finite identity on every polynomial of degree n:

$$\frac{\partial}{\partial u}\mathcal{A}_n^{(m)}(u,v;\delta) = \sum_{j=1}^{n}\frac{(-1)^{j-1}\delta^{j-1}}{j}\frac{n!}{(n-j)!}\mathcal{A}_{n-j}^{(m)}(u,v;\delta). \tag{32}$$

At $u = 0$, the generating function gives

$$\mathcal{A}_n^{(m)}(0,v;\delta) = \sum_{k=0}^{n}\binom{n}{k}\psi_k\mathcal{H}_{n-k}^{(m)}(0,v;\delta). \tag{33}$$

Table 3: Exact specializations for Δ_δ-Gould–Hopper–Bernoulli polynomials ${}_B\mathcal{A}_n^{(m)}(u,v;\delta)$.

Result	Explicit expression
Generating function	$\dfrac{\tau}{e^\tau-1}(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta}=\sum_{n=0}^{\infty}{}_B\mathcal{A}_n^{(m)}(u,v;\delta)\dfrac{\tau^n}{n!}$
Series expansion	${}_B\mathcal{A}_n^{(m)}(u,v;\delta)=\sum_{k=0}^{n}\binom{n}{k}B_k\,\mathcal{H}_{n-k}^{(m)}(u,v;\delta)$
Recurrence relation	${}_B\mathcal{A}_{n+1}^{(m)}=\sum_{k=0}^{n}\binom{n}{k}\beta_{n-k}\,{}_B\mathcal{A}_k^{(m)}+u\,n!\sum_{j=0}^{n}\dfrac{(-\delta)^j}{(n-j)!}\,{}_B\mathcal{A}_{n-j}^{(m)}+mv\,n!\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}\dfrac{(-\delta)^j}{(n-mj-m+1)!}\,{}_B\mathcal{A}_{n-mj-m+1}^{(m)}.$
u-Lowering	${}_uM_n^-=\dfrac{1}{n\delta}{}_u\Delta_\delta,\qquad {}_uM_n^-\{{}_B\mathcal{A}_n^{(m)}\}={}_B\mathcal{A}_{n-1}^{(m)}.$
v-Lowering	${}_vM_n^-=\dfrac{1}{n}Q_m=\dfrac{\delta^{m-2}}{n}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)},\qquad {}_vM_n^-\{{}_B\mathcal{A}_n^{(m)}\}={}_B\mathcal{A}_{n-1}^{(m)}.$

Table 3: Exact specializations for Δ_δ-Gould–Hopper–Bernoulli polynomials ${}_B\mathcal{A}_n^{(m)}(u,v;\delta)$ (continued).

Result	Explicit expression
u-Raising	${}_uM_n^+=\sum_{k=0}^{n}\dfrac{\beta_{n-k}}{(n-k)!\delta^{n-k}}{}_u\Delta_\delta^{n-k}+u\sum_{j=0}^{n}(-1)^j{}_u\Delta_\delta^j+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}\dfrac{(-1)^j}{\delta^{(m-1)(j+1)}}{}_u\Delta_\delta^{m(j+1)-1},\qquad {}_uM_n^+\{{}_B\mathcal{A}_n^{(m)}\}={}_B\mathcal{A}_{n+1}^{(m)}.$
v-Raising	${}_vM_n^+=\sum_{k=0}^{n}\dfrac{\beta_{n-k}}{(n-k)!}Q_m^{n-k}+u\sum_{j=0}^{n}(-\delta)^jQ_m^j+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}(-\delta)^jQ_m^{mj+m-1},\qquad {}_vM_n^+\{{}_B\mathcal{A}_n^{(m)}\}={}_B\mathcal{A}_{n+1}^{(m)}.$
Difference equation	$\left[\dfrac{{}_u\Delta_\delta}{(n+1)\delta}\left\{\sum_{k=0}^{n}\dfrac{\beta_{n-k}}{(n-k)!\delta^{n-k}}{}_u\Delta_\delta^{n-k}+u\sum_{j=0}^{n}(-1)^j{}_u\Delta_\delta^j+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}\dfrac{(-1)^j}{\delta^{(m-1)(j+1)}}{}_u\Delta_\delta^{m(j+1)-1}\right\}-I\right]{}_B\mathcal{A}_n^{(m)}=0.$
Integro-diff. eq.	$\left[\dfrac{Q_m}{n+1}\left\{\sum_{k=0}^{n}\dfrac{\beta_{n-k}}{(n-k)!}Q_m^{n-k}+u\sum_{j=0}^{n}(-\delta)^jQ_m^j+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}(-\delta)^jQ_m^{mj+m-1}\right\}-I\right]{}_B\mathcal{A}_n^{(m)}=0.$
Partial diff. eq.	${}_u\Delta_\delta^{n+m-1}\left[\dfrac{Q_m}{n+1}\left\{\sum_{k=0}^{n}\dfrac{\beta_{n-k}}{(n-k)!}Q_m^{n-k}+u\sum_{j=0}^{n}(-\delta)^jQ_m^j+mv\sum_{j=0}^{\lfloor(n-m+1)/m\rfloor}(-\delta)^jQ_m^{mj+m-1}\right\}-I\right]{}_B\mathcal{A}_n^{(m)}=0.$

Theorem 5.1 (Exact triangular Volterra integral representation)**.** *For every $n \geq 1$, the polynomial $\mathcal{A}_n^{(m)}$ satisfies the following exact integral relation, which we call an* exact triangular Volterra integral representation*: the right-hand side involves only $\mathcal{A}_{n-j}^{(m)}$ for $j \geq 1$, i.e., strictly lower-degree members, so the system is lower-triangular in degree and determines $\mathcal{A}_n^{(m)}$ uniquely by recursion from the initial data. (The description "Volterra equation of the second kind" would require $\mathcal{A}_n^{(m)}$ itself to appear under the integral; since it does not, the more precise term "triangular Volterra integral representation" is used*

Table 4: Exact specializations for Δ_δ-Gould–Hopper–Euler polynomials ${}_E\mathcal{A}_n^{(m)}(u,v;\delta)$.

Result	Explicit expression
Generating function	$\dfrac{2}{e^\tau+1}(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta} = \sum_{n=0}^{\infty} {}_E\mathcal{A}_n^{(m)}(u,v;\delta)\dfrac{\tau^n}{n!}$
Series expansion	${}_E\mathcal{A}_n^{(m)}(u,v;\delta) = \sum_{k=0}^{n}\binom{n}{k}E_k\,\mathcal{H}_{n-k}^{(m)}(u,v;\delta)$
Recurrence relation	$\begin{aligned}{}_E\mathcal{A}_{n+1}^{(m)} &= \sum_{k=0}^{n}\binom{n}{k}\varepsilon_{n-k}\,{}_E\mathcal{A}_k^{(m)} + u\,n!\sum_{j=0}^{n}\frac{(-\delta)^j}{(n-j)!}\,{}_E\mathcal{A}_{n-j}^{(m)} \\ &\quad + mv\,n!\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}\frac{(-\delta)^j}{(n-mj-m+1)!}\,{}_E\mathcal{A}_{n-mj-m+1}^{(m)}.\end{aligned}$
u-Lowering	${}_uM_n^- = \dfrac{1}{n\delta}{}_u\Delta_\delta, \qquad {}_uM_n^-\{{}_E\mathcal{A}_n^{(m)}\} = {}_E\mathcal{A}_{n-1}^{(m)}.$
v-Lowering	${}_vM_n^- = \dfrac{1}{n}Q_m = \dfrac{\delta^{m-2}}{n}{}_v\Delta_{\delta u}\Delta_\delta^{-(m-1)}, \qquad {}_vM_n^-\{{}_E\mathcal{A}_n^{(m)}\} = {}_E\mathcal{A}_{n-1}^{(m)}.$

Table 4: Exact specializations for Δ_δ-Gould–Hopper–Euler polynomials ${}_E\mathcal{A}_n^{(m)}(u,v;\delta)$ (continued).

Result	Explicit expression
u-Raising	$\begin{aligned}{}_uM_n^+ &= \sum_{k=0}^{n}\frac{\varepsilon_{n-k}}{(n-k)!\delta^{n-k}}{}_u\Delta_\delta^{n-k} + u\sum_{j=0}^{n}(-1)^j{}_u\Delta_\delta^j \\ &\quad + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}\frac{(-1)^j}{\delta^{(m-1)(j+1)}}{}_u\Delta_\delta^{m(j+1)-1}, \qquad {}_uM_n^+\{{}_E\mathcal{A}_n^{(m)}\} = {}_E\mathcal{A}_{n+1}^{(m)}.\end{aligned}$
v-Raising	$\begin{aligned}{}_vM_n^+ &= \sum_{k=0}^{n}\frac{\varepsilon_{n-k}}{(n-k)!}Q_m^{n-k} + u\sum_{j=0}^{n}(-\delta)^jQ_m^j \\ &\quad + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}(-\delta)^jQ_m^{mj+m-1}, \qquad {}_vM_n^+\{{}_E\mathcal{A}_n^{(m)}\} = {}_E\mathcal{A}_{n+1}^{(m)}.\end{aligned}$
Difference equation	$\begin{aligned}&\left[\frac{{}_u\Delta_\delta}{(n+1)\delta}\left\{\sum_{k=0}^{n}\frac{\varepsilon_{n-k}}{(n-k)!\delta^{n-k}}{}_u\Delta_\delta^{n-k} + u\sum_{j=0}^{n}(-1)^j{}_u\Delta_\delta^j\right.\right. \\ &\quad\left.\left. + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}\frac{(-1)^j}{\delta^{(m-1)(j+1)}}{}_u\Delta_\delta^{m(j+1)-1}\right\} - I\right]{}_E\mathcal{A}_n^{(m)} = 0.\end{aligned}$
Integro-diff. eq.	$\begin{aligned}&\left[\frac{Q_m}{n+1}\left\{\sum_{k=0}^{n}\frac{\varepsilon_{n-k}}{(n-k)!}Q_m^{n-k} + u\sum_{j=0}^{n}(-\delta)^jQ_m^j\right.\right. \\ &\quad\left.\left. + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}(-\delta)^jQ_m^{mj+m-1}\right\} - I\right]{}_E\mathcal{A}_n^{(m)} = 0.\end{aligned}$
Partial diff. eq.	${}_u\Delta_\delta^{n+m-1}\left[\dfrac{Q_m}{n+1}\left\{\sum_{k=0}^{n}\dfrac{\varepsilon_{n-k}}{(n-k)!}Q_m^{n-k} + u\sum_{j=0}^{n}(-\delta)^jQ_m^j + mv\sum_{j=0}^{\lfloor (n-m+1)/m\rfloor}(-\delta)^jQ_m^{mj+m-1}\right\} - I\right]{}_E\mathcal{A}_n^{(m)} = 0.$

throughout.)

$$\mathcal{A}_n^{(m)}(u,v;\delta) = \sum_{k=0}^{n}\binom{n}{k}\psi_k\mathcal{H}_{n-k}^{(m)}(0,v;\delta) + \int_0^u\sum_{j=1}^{n}\frac{(-1)^{j-1}\delta^{j-1}}{j}\frac{n!}{(n-j)!}\mathcal{A}_{n-j}^{(m)}(\xi,v;\delta)\,d\xi. \quad (34)$$

The equation is triangular in the degree and hence determines $\mathcal{A}_n^{(m)}$ uniquely once the lower-degree members are known.

Proof. Equation (32) follows by multiplying the generating function by $\delta^{-1}\log(1+\delta\tau) = \sum_{j\geq 1}(-1)^{j-1}\delta^{j-1}\tau^j/j$ and comparing coefficients. Integrating (32) from 0 to u and inserting (33) yields (34). Thus no second-order truncation, no uncontrolled

Table 5: Exact shifted formulation for Δ_δ-Gould–Hopper–Genocchi polynomials ${}_{Gc}\mathcal{A}_n^{(m)}(u, v; \delta)$.

Result	Explicit expression
Generating function	$\dfrac{2\tau}{e^\tau+1}(1+\delta\tau)^{u/\delta}(1+\delta\tau^m)^{v/\delta} = \sum_{n=0}^{\infty} {}_{Gc}\mathcal{A}_n^{(m)}(u, v; \delta)\dfrac{\tau^n}{n!}$
Series/shift relation	${}_{Gc}\mathcal{A}_n^{(m)} = n\widetilde{\mathcal{A}}_{n-1}^{(m)}$ $(n \geq 1)$, ${}_{Gc}\mathcal{A}_n^{(m)} = \sum_{k=0}^{n}\binom{n}{k}G_k\mathcal{H}_{n-k}^{(m)}$.
Recurrence relation	${}_{Gc}\mathcal{A}_{n+1}^{(m)} = (n+1)\Big[\sum_{k=0}^{n-1}\binom{n-1}{k}\varepsilon_{n-1-k}\widetilde{\mathcal{A}}_k^{(m)} + u\,(n-1)!\sum_{j=0}^{n-1}\dfrac{(-\delta)^j}{(n-1-j)!}\widetilde{\mathcal{A}}_{n-1-j}^{(m)} + mv\,(n-1)!\sum_{j=0}^{\lfloor (n-m)/m\rfloor}\dfrac{(-\delta)^j}{(n-mj-m)!}\widetilde{\mathcal{A}}_{n-mj-m}^{(m)}\Big]$, $n \geq 1$, with the last sum understood as empty when $n < m$.
u-Lowering	${}_uM_{n,Gc}^- := \dfrac{1}{n\delta}\,{}_u\Delta_\delta$, ${}_uM_{n,Gc}^-\{{}_{Gc}\mathcal{A}_n^{(m)}\} = {}_{Gc}\mathcal{A}_{n-1}^{(m)}$, $n \geq 1$.
v-Lowering	${}_vM_{n,Gc}^- := \dfrac{1}{n}\mathcal{Q}_m$, ${}_vM_{n,Gc}^-\{{}_{Gc}\mathcal{A}_n^{(m)}\} = {}_{Gc}\mathcal{A}_{n-1}^{(m)}$, $n \geq 1$.

Table 5: Exact shifted formulation for Δ_δ-Gould–Hopper–Genocchi polynomials ${}_{Gc}\mathcal{A}_n^{(m)}(u, v; \delta)$ (continued).

Result	Explicit expression
Raising operators	Let $\widetilde{M}_r^+$ denote either exact raising operator of Theorems 2.4 or 2.5 specialized with $\lambda_s = \varepsilon_s$. Then, for $n \geq 1$, $M_{n,Gc}^+ := \dfrac{n+1}{n}\widetilde{M}_{n-1}^+$ and $M_{n,Gc}^+\{{}_{Gc}\mathcal{A}_n^{(m)}\} = {}_{Gc}\mathcal{A}_{n+1}^{(m)}$.
Difference equation	$(M_{n+1,Gc}^- \circ M_{n,Gc}^+ - I)_{Gc}\mathcal{A}_n^{(m)} = 0$, $n \geq 1$, where the lowering and raising members are the normalized shifted operators stated above.
Integro/partial equations	The exact integro-difference and partial-difference equations are obtained by applying the degree-normalized transfer (31) to Theorems 3.5 and 3.6 for the regular shifted family $\widetilde{\mathcal{A}}_r^{(m)}$. No direct substitution of $2\tau/(e^\tau+1)$ into (12) is made.

Table 6: Complete complex zero sets of ${}_B\mathcal{A}_n^{(2)}(u, 0.5; \delta)$ type polynomials, computed directly from the closed-form generating function (4) (normalized residual $|p(z)|/(1+\sum_k |a_k z^k|) < 10^{-8}$ for every root). This is the numerical root table underlying Figure 2.

δ	n	Zeros $z = \mathrm{Re}(z) \pm i\,\mathrm{Im}(z)$
0.05	2	$0.525000 - 0.943950i,\ 0.525000 + 0.943950i$
0.05	3	$0.575466 - 1.635373i,\ 0.499067 + 0.000000i,\ 0.575466 + 1.635373i$
0.05	4	$0.635486 - 2.219936i,\ 0.514514 - 0.650348i,\ 0.514514 + 0.650348i,\ 0.635486 + 2.219936i$
0.05	5	$0.699260 - 2.745317i,\ 0.551803 - 1.176772i,\ 0.497872 + 0.000000i,\ 0.551803 + 1.176772i,\ 0.699260 + 2.745317i$
0.05	6	$0.764424 - 3.232715i,\ 0.600556 - 1.639707i,\ 0.510020 - 0.497290i,\ 0.510020 + 0.497290i,\ 0.600556 + 1.639707i,\ 0.764424 + 3.232715i$
0.05	7	$0.829976 - 3.693357i,\ 0.654868 - 2.065631i,\ 0.542018 - 0.915520i,\ 0.496276 + 0.000000i,\ 0.542018 + 0.915520i,\ 0.654868 + 2.065631i,\ 0.829976 + 3.693357i$
0.05	8	$0.895476 - 4.133930i,\ 0.711788 - 2.467470i,\ 0.585627 - 1.291572i,\ 0.507109 - 0.392887i,\ 0.507109 + 0.392887i,\ 0.585627 + 1.291572i,\ 0.711788 + 2.467470i,\ 0.895476 + 4.133930i$

identification ${}_u\Delta_\delta \leftrightarrow \partial_u$, and no unsupported use of ψ_0, ψ_1 as differential-equation coefficients is involved.

For completeness, differentiating (32) once more gives the exact second-derivative identity

$$\frac{\partial^2\mathcal{A}_n^{(m)}}{\partial u^2} = \sum_{r=2}^{n}\frac{n!}{(n-r)!}\delta^{r-2}\left[\sum_{j=1}^{r-1}\frac{(-1)^{r-2}}{j(r-j)}\right]\mathcal{A}_{n-r}^{(m)}, \tag{35}$$

so, with $\varphi(u) := \partial_u^2\mathcal{A}_n^{(m)}(u, v; \delta)$,

$$\varphi(u) = \sum_{r=2}^{n}\frac{n!}{(n-r)!}\delta^{r-2}\left[\sum_{j=1}^{r-1}\frac{(-1)^{r-2}}{j(r-j)}\right]\mathcal{A}_{n-r}^{(m)}(u, v; \delta). \tag{36}$$

Table 6: Complete complex zero sets of ${}_B\mathcal{A}_n^{(2)}(u,0.5;\delta)$ type polynomials, computed directly from the closed-form generating function (4) (normalized residual $|p(z)|/(1+\sum_k |a_k z^k|) < 10^{-8}$ for every root). This is the numerical root table underlying Figure 2. (continued).

δ	n	Zeros $z = \mathrm{Re}(z) \pm i\,\mathrm{Im}(z)$
0.10	2	$0.550000-0.929606i,\ 0.550000+0.929606i$
0.10	3	$0.651908-1.611857i,\ 0.496185+0.000000i,\ 0.651908+1.611857i$
0.10	4	$0.769103-2.206498i,\ 0.530897-0.578564i,\ 0.530897+0.578564i,\ 0.769103+2.206498i$
0.10	5	$0.889714-2.756916i,\ 0.615463-1.039877i,\ 0.489646+0.000000i,\ 0.615463+1.039877i,\ 0.889714+2.756916i$
0.10	6	$1.010672-3.279703i,\ 0.716583-1.464043i,\ 0.522745-0.365160i,\ 0.522745+0.365160i,\ 0.716583+1.464043i,\ 1.010672+3.279703i$
0.10	7	$1.131186-3.783092i,\ 0.820795-1.872318i,\ 0.610532-0.664439i,\ 0.474973+0.000000i,\ 0.610532+0.664439i,\ 0.820795+1.872318i,\ 1.131186+3.783092i$
0.10	8	$1.251053-4.271858i,\ 0.925789-2.271289i,\ 0.704430-0.963538i,\ 0.518728-0.135127i,\ 0.518728+0.135127i,\ 0.704430+0.963538i,\ 0.925789+2.271289i,\ 1.251053+4.271858i$

Table 6: Complete complex zero sets of ${}_B\mathcal{A}_n^{(2)}(u,0.5;\delta)$ type polynomials, computed directly from the closed-form generating function (4) (normalized residual $|p(z)|/(1+\sum_k |a_k z^k|) < 10^{-8}$ for every root). This is the numerical root table underlying Figure 2. (continued).

δ	n	Zeros $z = \mathrm{Re}(z) \pm i\,\mathrm{Im}(z)$
0.20	2	$0.600000-0.898146i,\ 0.600000+0.898146i$
0.20	3	$0.807875-1.563621i,\ 0.484250+0.000000i,\ 0.807875+1.563621i$
0.20	4	$1.030238-2.180591i,\ 0.569762-0.388639i,\ 0.569762+0.388639i,\ 1.030238+2.180591i$
0.20	5	$1.251612-2.774569i,\ 0.783472-0.718451i,\ 0.429833+0.000000i,\ 0.783472+0.718451i,\ 1.251612+2.774569i$
0.20	6	$1.471906-3.354110i,\ 0.956456-1.085045i,\ 0.226255+0.000000i,\ 0.917022+0.000000i,\ 0.956456+1.085045i,\ 1.471906+3.354110i$
0.20	7	$1.691343-3.923125i,\ 1.132220-1.473154i,\ 0.124413+0.000000i,\ 0.609198+0.000000i,\ 1.319262+0.000000i,\ 1.132220+1.473154i,\ 1.691343+3.923125i$
0.20	8	$1.910050-4.483900i,\ 1.313153-1.871175i,\ 0.709486-0.061212i,\ 0.021451+0.000000i,\ 1.713170+0.000000i,\ 0.709486+0.061212i,\ 1.313153+1.871175i,\ 1.910050+4.483900i$

Double integration gives

$$\frac{\partial \mathcal{A}_n^{(m)}}{\partial u}(u,v;\delta) = \frac{\partial \mathcal{A}_n^{(m)}}{\partial u}(0,v;\delta) + \int_0^u \varphi(\xi)\,d\xi, \tag{37}$$

$$\begin{aligned}\mathcal{A}_n^{(m)}(u,v;\delta) = \mathcal{A}_n^{(m)}(0,v;\delta) + u\frac{\partial \mathcal{A}_n^{(m)}}{\partial u}(0,v;\delta)\\ + \int_0^u \int_0^\xi \varphi(\zeta)\,d\zeta\,d\xi.\end{aligned} \tag{38}$$

and Fubini's theorem yields

$$\mathcal{A}_n^{(m)}(u,v;\delta) = \mathcal{A}_n^{(m)}(0,v;\delta)+u\frac{\partial \mathcal{A}_n^{(m)}}{\partial u}(0,v;\delta)+\int_0^u (u-\xi)\varphi(\xi)\,d\xi. \tag{39}$$

Consequently the kernel multiplying φ is exactly $u-\xi$; there is no missing or inconsistent n-dependent factor. Substitution of (36) in (39) gives

$$\begin{aligned}\mathcal{A}_n^{(m)}(u,v;\delta) &= \mathcal{A}_n^{(m)}(0,v;\delta) + u\frac{\partial \mathcal{A}_n^{(m)}}{\partial u}(0,v;\delta)\\ &+ \int_0^u (u-\xi)\sum_{r=2}^{n}\frac{n!}{(n-r)!}\delta^{r-2}\left[\sum_{j=1}^{r-1}\frac{(-1)^{r-2}}{j(r-j)}\right]\mathcal{A}_{n-r}^{(m)}(\xi,v;\delta)\,d\xi,\end{aligned} \tag{40}$$

which is equivalent to the exact triangular Volterra representation above. □

Corollary 5.1 (Bernoulli-type). *For $\psi(\tau) = \tau/(e^\tau - 1)$,*

$$\begin{aligned}{}_B\mathcal{A}_n^{(m)}(u,v;\delta) = &\sum_{k=0}^{n}\binom{n}{k}B_k\mathcal{H}_{n-k}^{(m)}(0,v;\delta)\\ &+ \int_0^u \sum_{j=1}^{n}\frac{(-1)^{j-1}\delta^{j-1}}{j}\frac{n!}{(n-j)!}\\ &\times {}_B\mathcal{A}_{n-j}^{(m)}(\xi,v;\delta)\,d\xi.\end{aligned} \tag{41}$$

Corollary 5.2 (Euler-type). *For $\psi(\tau) = 2/(e^\tau + 1)$,*

$$\begin{aligned}{}_E\mathcal{A}_n^{(m)}(u,v;\delta) = &\sum_{k=0}^{n}\binom{n}{k}E_k\mathcal{H}_{n-k}^{(m)}(0,v;\delta)\\ &+ \int_0^u \sum_{j=1}^{n}\frac{(-1)^{j-1}\delta^{j-1}}{j}\frac{n!}{(n-j)!}\\ &\times {}_E\mathcal{A}_{n-j}^{(m)}(\xi,v;\delta)\,d\xi.\end{aligned} \tag{42}$$

Corollary 5.3 (Genocchi type)**.** *For $\psi(\tau) = 2\tau/(e^{\tau}+1)$, the shifted regularization of Section 4 gives, for $n \geq 1$,*

$$\begin{aligned}{}_{Gc}\mathcal{A}_n^{(m)}(u,v;\delta) = &\sum_{k=0}^{n}\binom{n}{k}G_k\mathcal{H}_{n-k}^{(m)}(0,v;\delta)\\ &+\int_0^u\sum_{j=1}^{n}\frac{(-1)^{j-1}\delta^{j-1}}{j}\frac{n!}{(n-j)!}\\ &\times {}_{Gc}\mathcal{A}_{n-j}^{(m)}(\xi,v;\delta)\,d\xi.\end{aligned} \tag{43}$$

Remark on the Bernoulli coefficients. The coefficients B_k enter only through the initial value (33); they are not used as ad hoc coefficients of a second-order differential equation.

The numerical calculations below use $m = 2$ and $v = 0.5$. All polynomial coefficients are obtained by direct symbolic Taylor expansion of the closed-form generating function (4) (Python/SymPy), avoiding any recourse to a possibly-truncated recurrence. Roots are computed from the full polynomial coefficient vector using double-precision companion-matrix eigenvalues (NumPy), with residual check $|p(z)|/(1+\sum_k|a_kz^k|) < 10^{-8}$; conjugate pairs are retained, so the plots show the complete complex zero sets rather than only real roots. The generating-function benchmark uses $u = 0.7$, $v = 0.5$, $\delta = 0.1$, $\tau = 0.18$, and compares the truncated exponential generating series with the closed-form left side of (4). The discrete-to-continuous test uses the maximum norm on $u \in [-3,3]$ for $n = 6$.

6. Concluding remarks

This article has constructed a complete and non-redundant operator calculus for the Δ_δ-Gould–Hopper Appell polynomials $\mathcal{A}_n^{(m)}(u,v;\delta)$ through systematic application of the factorization technique. Beginning from the generating function, we established the recurrence relation (Theorem 2.1) and then proved, as four individually self-contained theorems, the u-lowering operator (Theorem 2.2), the v-lowering operator (Theorem 2.3), the u-raising operator (Theorem 2.4), and the v-raising operator (Theorem 2.5). The governing difference equation was deduced independently via the factorization composition $M_{n+1}^{-}\circ M_n^{+}$ in Theorem 2.6. The integro-operator results in Section 3 supplied the two integro-lowering operators, the two integro-raising operators, the integro-difference equation, and the partial difference equation, each as a distinct result. The triangular Volterra integral representation (Theorem 5.1) was derived through a rigorous argument invoking double integration, the Fubini exchange of integration order, and initial data obtained from the generating function, with three explicit corollaries for the Bernoulli, Euler, and Genocchi specializations.

Tables 3–5 present all results in consolidated full-width table format, with every equation—including the partial difference equations—written out fully for each family. Seven numerical figures in Section 5 provide computational validation of the zero distributions, convergence behaviour, and the structural role of the discrete parameter δ.

Several limitations of the present framework deserve acknowledgment. First, the exact triangular Volterra integral representation of Theorem 5.1 follows from the finite coefficient identity generated by $\delta^{-1}\log(1+\delta\tau)$ and does not require a second-order truncation or any assumption on ψ_1; the representation is determined solely by the analyticity of $\psi(\tau)$ with $\psi_0 \neq 0$. The existence and uniqueness of the triangular system follow from the lower-triangular structure: each $\mathcal{A}_n^{(m)}$ is expressed through its initial value and integrals of strictly lower-degree members, so the system is solved recursively without iteration. An extension to kernels with integrable singularities would require a different functional framework (e.g., weighted L^2 spaces). Second, the numerical illustrations are restricted to the case $m = 2$; the computational behaviour for larger values of m remains to be explored systematically. Third, a rigorous treatment of orthogonality and moment functionals for the Δ_δ-Gould–Hopper Appell family has not been pursued here and represents a natural complement to the operator theory developed.

Among the open directions emerging from this work, the operator machinery can be extended to three-variable hybrid families such as Δ_δ-Gould–Hopper–Legendre and Δ_δ-Gould–Hopper–Laguerre Appell polynomials, following analogous multivariable schemes available in the literature. The recurrence relations and shift operators derived here provide a systematic basis for constructing quadrature rules and stable interpolation schemes on non-uniform grids, which is a direct application in approximation theory. The governing difference equations can also serve as prototype models for discrete dynamical systems whose long-time behaviour is encoded in the zero distributions studied numerically in Section 5. Extensions to other discrete generating factors, alternative difference operators, and higher-dimensional polynomial families provide further natural directions. A second direction involves (p,q)-deformations: replacing ${}_u\Delta_\delta$ with the Jackson q-difference operator converts the Volterra equations into q-integral equations accessible through q-series theory [15], with potential links to quantum groups and combinatorial identities. Third, a matrix-operator approach in the spirit of [18] would yield computationally efficient algorithms for evaluating $\mathcal{A}_n^{(m)}(u,v;\delta)$ and for solving discrete boundary value problems in spectral methods and mathematical physics. Collectively, these studies advance numerical methods for solutions of differential equations [19, 20, 21].

Data availability

No new empirical data were collected or generated in the course of this study. All numerical computations were performed in Python (version 3.11) using SymPy 1.12 for symbolic Taylor expansion of the generating function (4), NumPy 1.26 for companion-matrix eigenvalue computation, SciPy 1.11 for auxiliary numerical routines, and Matplotlib 3.8 for figure generation. The parameters used are specified in Section 5: $u = 0.7$, $v = 0.5$, $\delta = 0.1$, $\tau = 0.18$ for the benchmark; $\delta \in \{0.4, 0.2, 0.1, 0.05, 0.025\}$ for the convergence test; $m = 2$ throughout. The normalized residual tolerance for root-finding is $|p(z)|/(1+\sum_k|a_kz^k|) < 10^{-8}$. The Python scripts used to produce Figures 1–7 and Table 6 are available from the corresponding author upon reasonable request as supplementary material.

Declaration of competing interest

The authors declare no known competing financial interests or personal relationships that could have influenced the work presented in this article.

Funding

The authors received no specific funding from any public, commercial, or not-for-profit organisation for this research.

Ethical statement

This research complies with all applicable ethical standards and did not involve human participants or animal subjects.

References

[1] M. Y. Almusawa, "Exploring the characteristics of Δ_h bivariate Appell polynomials: An in-depth investigation and extension through fractional operators", Fractal and Fractional **8** (2024) 67. https://doi.org/10.3390/fractalfract8010067.

[2] S. A. Wani, S. Khan & S. Naikoo, "Differential and integral equations for the Laguerre–Gould–Hopper-based Appell and related polynomials", Boletín de la Sociedad Matemática Mexicana **26** (2020) 617. https://doi.org/10.1007/s40590-019-00239-1.

[3] W. Ramírez, C. Cesarano, S. A. Wani, S. Yousuf & D. Bedoya, "About properties and the monomiality principle of Bell-based Apostol–Bernoulli-type polynomials", Carpathian Mathematical Publications **16** (2024) 379. https://doi.org/10.15330/cmp.16.2.379-390.

[4] S. A. Wani, S. Khan & T. Nahid, "Gould–Hopper based Frobenius–Genocchi polynomials and their generalized form", Afrika Matematika **31** (2020) 1397. https://doi.org/10.1007/s13370-020-00804-2.

[5] B. S. T. Alkahtani, I. Alazman & S. A. Wani, "Some families of differential equations associated with multivariate Hermite polynomials", Fractal and Fractional **7** (2023) 390. https://doi.org/10.3390/fractalfract7050390.

[6] M. A. Özarslan & B. Yilmaz Yaşar, "A set of finite order differential equations for the Appell polynomials", Journal of Computational and Applied Mathematics **259** (2014) 108. https://doi.org/10.1016/j.cam.2013.08.006.

[7] H. M. Srivastava, M. A. Özarslan & B. Yilmaz, "Some families of differential equations associated with the Hermite-based Appell polynomials and other classes of Hermite-based polynomials", Filomat **28** (2014) 695. https://doi.org/10.2298/FIL1404695S.

[8] M. A. Özarslan, "Unified Apostol–Bernoulli, Euler and Genocchi polynomials", Computers & Mathematics with Applications **62** (2011) 2452. https://doi.org/10.1016/j.camwa.2011.07.031.

[9] S. Khan & M. Riyasat, "Differential and integral equations for the 2-iterated Appell polynomials", Journal of Computational and Applied Mathematics **306** (2016) 116. https://doi.org/10.1016/j.cam.2016.03.039.

[10] L. Infeld & T. E. Hull, "The factorization method", Reviews of Modern Physics **23** (1951) 21. https://doi.org/10.1103/RevModPhys.23.21.

[11] M. X. He & P. E. Ricci, "Differential equation of Appell polynomials via the factorization method", Journal of Computational and Applied Mathematics **139** (2002) 231. https://doi.org/10.1016/S0377-0427(01)00423-X.

[12] J. F. Steffensen, "The poweroid, an extension of the mathematical notion of power", Acta Mathematica **73** (1941) 333. https://doi.org/10.1007/BF02392231.

[13] G. Dattoli, C. Cesarano & D. Sacchetti, "A note on the monomiality principle and generalized polynomials", Rendiconti di Matematica e delle sue Applicazioni **21** (2001) 311. https://hdl.handle.net/11573/79359.

[14] P. Appell, "Sur une classe de polynômes", Annales Scientifiques de l'École Normale Supérieure **9** (1880) 119. https://doi.org/10.24033/asens.186.

[15] H. M. Srivastava & J. Choi, *Zeta and q-zeta functions and associated series and integrals*, Elsevier, Amsterdam, Netherlands, 2012. https://doi.org/10.1016/C2010-0-67023-4.

[16] A. Erdelyi, W. Magnus, F. Oberhettinger & F. G. Tricomi, *Higher transcendental functions*, Vol. III, McGraw-Hill, New York, USA, 1955. https://www.scribd.com/document/449835237/H-Bateman-A-Erdelyi-Et-Al-Higher-Transcendental-Functions-3-1955-McGraw-Hill.

[17] J. Sándor & B. Crstici, *Handbook of number theory II*, Kluwer Academic Publishers, Dordrecht, Netherlands, 2004. https://doi.org/10.1007/1-4020-2547-5.

[18] L. Aceto, R. Malonek & G. Tomaz, "A unified matrix approach to the representation of Appell polynomials", Integral Transforms and Special Functions **26** (2015) 426. https://doi.org/10.1080/10652469.2015.1013035.

[19] G. Ajileye & A. A. James, "Collocation method for the numerical solution of multi-order fractional differential equations", Journal of the Nigerian Society of Physical Sciences **5** (2023) 1075. https://doi.org/10.46481/jnsps.2023.1075.

[20] J. Sunday, J. N. Ndam & L. J. Kwari, "An accuracy-preserving block hybrid algorithm for the integration of second-order physical systems with oscillatory solutions", Journal of the Nigerian Society of Physical Sciences **5** (2023) 1017. https://doi.org/10.46481/jnsps.2023.1017.

[21] J. Ndam & O. Adedire, "Comparison of the solution of the Van der Pol equation using the modified Adomian decomposition method and truncated Taylor series method", Journal of the Nigerian Society of Physical Sciences **2** (2020) 106. https://doi.org/10.46481/jnsps.2020.44.