



Quantum trapezoid-type inequalities for coordinated strongly s -convex functions

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Abstract

We establish new q -trapezoid-type inequalities for q -differentiable coordinated strongly s -convex functions on bidimensional intervals. A symmetric modulus convention is adopted for the two coordinates, and the main estimates are derived using a quantum integral identity, the q -Hölder inequality, and the q -power mean inequality. The corresponding classical inequalities are obtained by letting $q_1, q_2 \rightarrow 1^-$. Explicit examples verify the required strong s -convexity assumptions and illustrate the applicability of the results. Numerical computations on a specified finite grid, together with graphical illustrations, support the theoretical estimates.

DOI: [10.46481/jnsps.2027.3670](https://doi.org/10.46481/jnsps.2027.3670)

Keywords: Quantum calculus, Trapezoid-type inequalities, Coordinated strong s -convexity, q -integrals, Convexity

Article history:

Received: 05 July 2026

Received in revised form: 04 September 2026

Accepted for publication: 22 September 2026

Available online: 21 October 2026

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Communicated by: P. Thakur

1. Introduction

Convexity is a fundamental concept in mathematical analysis with applications in optimization, engineering, physics, information theory, and inequality theory. Its importance has led to several generalizations that extend classical results to broader classes of functions in Refs. [1, 2]. Related developments and applications can be found in Refs. [3, 4].

Strong convexity refines ordinary convexity by introducing a positive modulus. Karamardian in Ref. [5] established a basic connection between strong convexity and strong monotonicity. For functions of two variables, Dragomir in Ref. [6] introduced convexity on the coordinates and showed that every convex function on a rectangle is coordinated convex, while

the converse does not generally hold. Adil *et al.* in Ref. [7] subsequently introduced coordinated strongly convex functions and established corresponding Slater-, Jensen-, and Hermite-Hadamard-type inequalities. Further results in this direction were obtained in Refs. [8–10].

Ullah *et al.* in Ref. [11] extended this framework by introducing coordinated strongly s -convex functions, where $s \in (0, 1]$. This class reduces to coordinated strongly convex functions when $s = 1$ and provides a broader setting for the study of integral inequalities for functions of two variables.

Quantum calculus, or q -calculus, concerns analogues of classical differentiation and integration without using limits. The q -derivative and q -integral introduced by Jackson in Ref. [12] recover their classical counterparts as $q \rightarrow 1^-$. These operators have been applied to various problems in analysis, particularly integral inequalities and differential subordination (Refs. [13–17]). Silviya and Muthunagai in Ref. [18], for example, studied differential and fuzzy differential sandwich the-

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orems involving quantum calculus operators. Quantum-based mathematical methods also arise in other areas, including quantum computing (Ref. [19]). The present work is concerned specifically with Jackson's q -difference and q -integral operators.

Quantum integral inequalities for functions of two variables have received considerable attention. Trapezoid- and Simpson-type estimates for coordinated convex functions were developed in Refs. [20, 21], while quantum Ostrowski-type inequalities and related Montgomery identities were studied in Refs. [22–24]. Further extensions under generalized convexity assumptions were obtained by You *et al.* in Ref. [25], Kalsoom *et al.* in Refs. [26, 27], and Zhang *et al.* in Ref. [28]. Closely related to the present study, Mishra *et al.* in Ref. [29] established quantum trapezoid-type inequalities for coordinated strongly convex functions.

Motivated by these developments, we extend the quantum trapezoid-type framework to coordinated strongly s -convex functions. A symmetric modulus convention is adopted for the two coordinates, consistently with the characterization in terms of partial mappings. Starting from a mixed quantum integral identity, we derive three estimates using direct estimation, the quantum Hölder inequality, and the quantum power mean inequality. Their classical counterparts are obtained by taking $q_1, q_2 \rightarrow 1^-$. We also provide examples in which the required coordinated strong s -convexity assumptions are verified explicitly, together with numerical and graphical illustrations of the resulting bounds.

The remainder of the paper is organized as follows. Section 2 reviews the definitions and auxiliary results required for the subsequent analysis. Section 3 presents a mixed quantum integral identity, the main trapezoid-type inequalities, and their classical limits. Section 4 provides explicit examples, numerical computations, and graphical illustrations. Finally, Section 5 summarizes the main findings.

2. Preliminaries

This section presents the basic definitions and auxiliary results used in the sequel. Throughout the paper, let

$$\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$$

be a rectangle, and let

$$\Delta^\circ = (a_1, b_1) \times (a_2, b_2)$$

denote its interior. We assume that

$$q, q_1, q_2 \in (0, 1).$$

2.1. Convexity notions

We first recall the classes of convex functions required in the main results.

Definition 2.1. [30] A function $\Psi : [a_1, b_1] \rightarrow \mathbb{R}$ is said to be *convex* if

$$\Psi(\tau u + (1 - \tau)v) \leq \tau\Psi(u) + (1 - \tau)\Psi(v)$$

for all $u, v \in [a_1, b_1]$ and $\tau \in [0, 1]$.

Definition 2.2. [31] Let $c > 0$. A function $\Psi : [a_1, b_1] \rightarrow \mathbb{R}$ is said to be *strongly convex with modulus c* if

$$\Psi(\tau u + (1 - \tau)v) \leq \tau\Psi(u) + (1 - \tau)\Psi(v) - c\tau(1 - \tau)(u - v)^2$$

for all $u, v \in [a_1, b_1]$ and $\tau \in [0, 1]$.

Definition 2.3. [32] Let $c > 0$ and $s \in (0, 1]$. A function $\Psi : [a_1, b_1] \rightarrow [0, \infty)$ is said to be *strongly s -convex with modulus c* if

$$\Psi(\tau u + (1 - \tau)v) \leq \tau^s\Psi(u) + (1 - \tau)^s\Psi(v) - c\tau(1 - \tau)(u - v)^2$$

for all $u, v \in [a_1, b_1]$ and $\tau \in [0, 1]$.

Definition 2.4. [6] A function $\Psi : \Delta \rightarrow \mathbb{R}$ is said to be *coordinated convex* if, for every fixed $v \in [a_2, b_2]$, the partial mapping

$$\Psi_v(u) = \Psi(u, v), \quad u \in [a_1, b_1],$$

is convex on $[a_1, b_1]$, and, for every fixed $u \in [a_1, b_1]$, the partial mapping

$$\Psi_u(v) = \Psi(u, v), \quad v \in [a_2, b_2],$$

is convex on $[a_2, b_2]$. Equivalently, Ψ is coordinated convex if

$$\begin{aligned} &\Psi(\tau u_1 + (1 - \tau)u_2, \xi v_1 + (1 - \xi)v_2) \\ &\leq \tau\xi\Psi(u_1, v_1) + \tau(1 - \xi)\Psi(u_1, v_2) \\ &\quad + (1 - \tau)\xi\Psi(u_2, v_1) + (1 - \tau)(1 - \xi)\Psi(u_2, v_2) \end{aligned}$$

for all $u_1, u_2 \in [a_1, b_1]$, $v_1, v_2 \in [a_2, b_2]$, and $\tau, \xi \in [0, 1]$.

Following Ref. [7], we recall coordinated strong convexity using the moduli of the two partial mappings.

Definition 2.5. Let $c_1, c_2 > 0$. A function $\Psi : \Delta \rightarrow \mathbb{R}$ is said to be *coordinated strongly convex with respect to (c_1, c_2)* if, for every fixed $v \in [a_2, b_2]$, the partial mapping

$$\Psi_v(u) = \Psi(u, v), \quad u \in [a_1, b_1],$$

is strongly convex with modulus c_1 on $[a_1, b_1]$, and, for every fixed $u \in [a_1, b_1]$, the partial mapping

$$\Psi_u(v) = \Psi(u, v), \quad v \in [a_2, b_2],$$

is strongly convex with modulus c_2 on $[a_2, b_2]$.

The preceding definition yields the following symmetric four-point inequality:

$$\begin{aligned} &\Psi(\tau u_1 + (1 - \tau)u_2, \xi v_1 + (1 - \xi)v_2) \\ &\leq \tau\xi\Psi(u_1, v_1) + \tau(1 - \xi)\Psi(u_1, v_2) \\ &\quad + (1 - \tau)\xi\Psi(u_2, v_1) + (1 - \tau)(1 - \xi)\Psi(u_2, v_2) \\ &\quad - c_1\tau(1 - \tau)(u_1 - u_2)^2 - c_2\xi(1 - \xi)(v_1 - v_2)^2, \end{aligned}$$

for all $u_1, u_2 \in [a_1, b_1]$, $v_1, v_2 \in [a_2, b_2]$, and $\tau, \xi \in [0, 1]$.

Following Ullah *et al.* in Ref. [11], we recall the corresponding s -convex class.

Definition 2.6. Let $c_1, c_2 > 0$ and $s \in (0, 1]$. A function $\Psi : \Delta \rightarrow [0, \infty)$ is said to be *coordinated strongly s -convex with respect to (c_1, c_2)* if the following conditions hold:

1. For every fixed $v \in [a_2, b_2]$, the partial mapping

$$\Psi_v(u) = \Psi(u, v), \quad u \in [a_1, b_1],$$

is strongly s -convex with modulus c_1 on $[a_1, b_1]$.

2. For every fixed $u \in [a_1, b_1]$, the partial mapping

$$\Psi_u(v) = \Psi(u, v), \quad v \in [a_2, b_2],$$

is strongly s -convex with modulus c_2 on $[a_2, b_2]$.

The corresponding four-point estimate is given below.

Proposition 2.1. Let $\Psi : \Delta \rightarrow [0, \infty)$ be coordinated strongly s -convex with respect to (c_1, c_2) . Then

$$\begin{aligned} & \Psi(\tau u_1 + (1 - \tau)u_2, \xi v_1 + (1 - \xi)v_2) \\ & \leq \tau^s \xi^s \Psi(u_1, v_1) + \tau^s (1 - \xi)^s \Psi(u_1, v_2) \\ & \quad + (1 - \tau)^s \xi^s \Psi(u_2, v_1) + (1 - \tau)^s (1 - \xi)^s \Psi(u_2, v_2) \\ & \quad - c_1 \tau(1 - \tau)(u_1 - u_2)^2 - c_2 \xi(1 - \xi)(v_1 - v_2)^2, \end{aligned} \quad (2.1)$$

for all $u_1, u_2 \in [a_1, b_1]$, $v_1, v_2 \in [a_2, b_2]$, and $\tau, \xi \in [0, 1]$.

Proof. Set

$$u_\tau = \tau u_1 + (1 - \tau)u_2, \quad v_\xi = \xi v_1 + (1 - \xi)v_2.$$

By the strong s -convexity of the partial mapping in the first coordinate,

$$\begin{aligned} \Psi(u_\tau, v_\xi) & \leq \tau^s \Psi(u_1, v_\xi) + (1 - \tau)^s \Psi(u_2, v_\xi) \\ & \quad - c_1 \tau(1 - \tau)(u_1 - u_2)^2. \end{aligned} \quad (2.2)$$

Applying strong s -convexity in the second coordinate gives

$$\begin{aligned} \Psi(u_1, v_\xi) & \leq \xi^s \Psi(u_1, v_1) + (1 - \xi)^s \Psi(u_1, v_2) \\ & \quad - c_2 \xi(1 - \xi)(v_1 - v_2)^2 \end{aligned}$$

and

$$\begin{aligned} \Psi(u_2, v_\xi) & \leq \xi^s \Psi(u_2, v_1) + (1 - \xi)^s \Psi(u_2, v_2) \\ & \quad - c_2 \xi(1 - \xi)(v_1 - v_2)^2. \end{aligned}$$

Substituting these estimates into (2.2), we obtain

$$\begin{aligned} \Psi(u_\tau, v_\xi) & \leq \tau^s \xi^s \Psi(u_1, v_1) + \tau^s (1 - \xi)^s \Psi(u_1, v_2) \\ & \quad + (1 - \tau)^s \xi^s \Psi(u_2, v_1) + (1 - \tau)^s (1 - \xi)^s \Psi(u_2, v_2) \\ & \quad - c_1 \tau(1 - \tau)(u_1 - u_2)^2 \\ & \quad - c_2 [\tau^s + (1 - \tau)^s] \xi(1 - \xi)(v_1 - v_2)^2. \end{aligned}$$

Since $s \in (0, 1]$,

$$\tau^s + (1 - \tau)^s \geq 1.$$

Consequently,

$$\begin{aligned} & -c_2 [\tau^s + (1 - \tau)^s] \xi(1 - \xi)(v_1 - v_2)^2 \\ & \leq -c_2 \xi(1 - \xi)(v_1 - v_2)^2. \end{aligned}$$

This proves (2.1). $\square$

Remark 2.1. The displayed four-point formulas in Refs. [7, 11] contain an additional factor 2 in the penalty term associated with the second coordinate. In the present paper, c_1 and c_2 denote the strong s -convexity moduli of the partial mappings in the first and second coordinates, respectively. Accordingly, the symmetric penalty terms

$$-c_1 \tau(1 - \tau)(u_1 - u_2)^2 - c_2 \xi(1 - \xi)(v_1 - v_2)^2$$

are used throughout the paper.

2.2. Basic q -calculus

In this subsection, we briefly review several fundamental concepts of q -calculus that will be needed in the subsequent analysis. Let $q, q_1, q_2 \in (0, 1)$ and $\Delta = [a_1, b_1] \times [a_2, b_2]$. For convenience, we use the following notation throughout the paper:

$$[\kappa]_q := \begin{cases} \frac{1 - q^\kappa}{1 - q}, & \text{if } \kappa \in \mathbb{R}^+, \\ 0, & \text{if } \kappa = 0. \end{cases}$$

For $\kappa \in \mathbb{N}$, this reduces to

$$[\kappa]_q = 1 + q + \cdots + q^{\kappa-1}.$$

Definition 2.7. The ${}_a q$ -derivative of a function $\Psi : [a, b] \xrightarrow{\text{cont.}} \mathbb{R}$ at $u \in [a, b]$ is defined by

$${}_a D_q \Psi(u) = \frac{\Psi(u) - \Psi(qu + (1 - q)a)}{(1 - q)(u - a)}, \quad u \neq a. \quad (2.3)$$

If $u = a$, we define

$${}_a D_q \Psi(a) = \lim_{u \rightarrow a} {}_a D_q \Psi(u),$$

provided that the limit exists and is finite; see Refs. [16, 33].

When $a = 0$, equation (2.3) reduces to

$${}_0 D_q \Psi(u) = \frac{\Psi(u) - \Psi(qu)}{(1 - q)u}, \quad u \neq 0,$$

which coincides with the classical q -Jackson derivative; see Refs. [12, 34]. We next introduce the corresponding right-sided q -derivative.

Definition 2.8. The ${}_b q$ -derivative of a function $\Psi : [a, b] \xrightarrow{\text{cont.}} \mathbb{R}$ at $u \in [a, b]$ is defined by

$${}_b D_q \Psi(u) = \frac{\Psi(qu + (1 - q)b) - \Psi(u)}{(1 - q)(b - u)}, \quad u \neq b. \quad (2.4)$$

If $u = b$, we define

$${}_b D_q \Psi(b) = \lim_{u \rightarrow b} {}_b D_q \Psi(u),$$

provided that the limit exists and is finite; see Ref. [15].

We now define the associated q -integrals.

Definition 2.9. The ${}_a q$ -integral of a function $\Psi : [a, b] \xrightarrow[\text{cont.}]{} \mathbb{R}$ from a to $u \in [a, b]$ is defined by

$$\int_a^u \Psi(t) {}_a d_q t = (1-q)(u-a) \sum_{\gamma=0}^{\infty} q^\gamma \Psi(q^\gamma u + (1-q^\gamma)a). \quad (2.5)$$

When $a = 0$, equation (2.5) reduces to

$$\int_0^u \Psi(t) d_q t = (1-q)u \sum_{\gamma=0}^{\infty} q^\gamma \Psi(q^\gamma u),$$

which corresponds to the classical q -Jackson integral; see Refs. [16, 33].

Definition 2.10. The ${}_b q$ -integral of a function $\Psi : [a, b] \xrightarrow[\text{cont.}]{} \mathbb{R}$ from $u \in [a, b]$ to b is defined by

$$\int_u^b \Psi(t) {}^b d_q t = (1-q)(b-u) \sum_{\gamma=0}^{\infty} q^\gamma \Psi(q^\gamma u + (1-q^\gamma)b). \quad (2.6)$$

When $u = 0$, equation (2.6) becomes

$$\int_0^b \Psi(t) {}^b d_q t = (1-q)b \sum_{\gamma=0}^{\infty} q^\gamma \Psi((1-q^\gamma)b),$$

see Ref. [15].

2.3. Two-variable quantum integrals

Quantum integrals for functions of two variables were first introduced by Latif *et al.* in Ref. [35] and subsequently studied by Budak *et al.* in Ref. [36].

Definition 2.11. [35] Let $\Psi : \Delta \rightarrow \mathbb{R}$ be a function of two variables. Then, for $(u, v) \in \Delta$, the quantum integral is defined by

$$\begin{aligned} & \int_{a_1}^u \int_{a_2}^v \Psi(s, t) {}_{a_2} d_{q_2} t {}_{a_1} d_{q_1} s \\ &= (1-q_1)(1-q_2)(u-a_1)(v-a_2) \\ & \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} q_1^k q_2^l \Psi(q_1^k u + (1-q_1^k)a_1, q_2^l v + (1-q_2^l)a_2). \end{aligned}$$

Definition 2.12. [36] Let $\Psi : \Delta \rightarrow \mathbb{R}$ be a function of two variables. Then, for $(u, v) \in \Delta$, the following quantum integrals are defined by

$$\begin{aligned} & \int_{a_1}^u \int_v^{b_2} \Psi(s, t) {}^{b_2} d_{q_2} t {}_{a_1} d_{q_1} s \\ &= (1-q_1)(1-q_2)(u-a_1)(b_2-v) \\ & \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} q_1^k q_2^l \Psi(q_1^k u + (1-q_1^k)a_1, q_2^l v + (1-q_2^l)b_2), \end{aligned}$$

$$\int_u^{b_1} \int_{a_2}^v \Psi(s, t) {}_{a_2} d_{q_2} t {}^{b_1} d_{q_1} s$$

$$\begin{aligned} &= (1-q_1)(1-q_2)(b_1-u)(v-a_2) \\ & \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} q_1^k q_2^l \Psi(q_1^k u + (1-q_1^k)b_1, q_2^l v + (1-q_2^l)a_2), \end{aligned}$$

and

$$\begin{aligned} & \int_u^{b_1} \int_v^{b_2} \Psi(s, t) {}^{b_2} d_{q_2} t {}^{b_1} d_{q_1} s \\ &= (1-q_1)(1-q_2)(b_1-u)(b_2-v) \\ & \times \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} q_1^k q_2^l \Psi(q_1^k u + (1-q_1^k)b_1, q_2^l v + (1-q_2^l)b_2). \end{aligned}$$

The following q_1, q_2 -derivatives for functions of two variables were introduced by Latif *et al.* in Ref. [35] and later extended by Budak *et al.* in Ref. [36].

Definition 2.13. [35] Let $\Psi : \Delta \xrightarrow[\text{cont.}]{} \mathbb{R}$ be a continuous function of two variables, where $0 < q_1 < 1$ and $0 < q_2 < 1$. The partial q_1 -derivative, q_2 -derivative, and $q_1 q_2$ -derivative of Ψ at $(u, v) \in \Delta$ are defined as follows:

$$\begin{aligned} \frac{{}_{a_1} \partial_{q_1} \Psi(u, v)}{{}_{a_1} \partial_{q_1} u} &= \frac{\Psi(u, v) - \Psi(q_1 u + (1-q_1)a_1, v)}{(1-q_1)(u-a_1)}, \quad u \neq a_1. \\ \frac{{}_{a_2} \partial_{q_2} \Psi(u, v)}{{}_{a_2} \partial_{q_2} v} &= \frac{\Psi(u, v) - \Psi(u, q_2 v + (1-q_2)a_2)}{(1-q_2)(v-a_2)}, \quad v \neq a_2, \end{aligned}$$

and

$$\begin{aligned} & \frac{{}_{a_1, a_2} \partial_{q_1, q_2}^2 \Psi(u, v)}{{}_{a_1} \partial_{q_1} u {}_{a_2} \partial_{q_2} v} \\ &= \frac{1}{(1-q_1)(1-q_2)(u-a_1)(v-a_2)} \\ & \times [\Psi(q_1 u + (1-q_1)a_1, q_2 v + (1-q_2)a_2) \\ & - \Psi(q_1 u + (1-q_1)a_1, v) - \Psi(u, q_2 v \\ & + (1-q_2)a_2) + \Psi(u, v)], \quad u \neq a_1, v \neq a_2. \end{aligned}$$

The function $\Psi : \Delta \xrightarrow[\text{cont.}]{} \mathbb{R}$ is said to be partially q_1 -, q_2 -, and $q_1 q_2$ -differentiable on Δ if

$$\frac{{}_{a_1} \partial_{q_1} \Psi(u, v)}{{}_{a_1} \partial_{q_1} u}, \quad \frac{{}_{a_2} \partial_{q_2} \Psi(u, v)}{{}_{a_2} \partial_{q_2} v}, \quad \frac{{}_{a_1, a_2} \partial_{q_1, q_2}^2 \Psi(u, v)}{{}_{a_1} \partial_{q_1} u {}_{a_2} \partial_{q_2} v},$$

exist for all $(u, v) \in \Delta$.

2.4. Mixed quantum partial derivatives

Following Ref. [36], we introduce the one-sided and mixed second-order partial q -derivatives used throughout this paper.

Definition 2.14. [36] Let

$$\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2,$$

let $\Psi \in C(\Delta)$, and let $q_1, q_2 \in (0, 1)$. The right-sided partial q_1 -derivative of Ψ with respect to the first variable is defined by

$$\frac{{}^{b_1} \partial_{q_1} \Psi(u, v)}{{}^{b_1} \partial_{q_1} u} = \frac{\Psi(q_1 u + (1-q_1)b_1, v) - \Psi(u, v)}{(1-q_1)(b_1-u)}, \quad u \neq b_1.$$

Similarly, the right-sided partial q_2 -derivative of Ψ with respect to the second variable is defined by

$$\frac{{}^{b_2}\partial_{q_2}\Psi(u, v)}{{}^{b_2}\partial_{q_2}v} = \frac{\Psi(u, q_2v + (1 - q_2)b_2) - \Psi(u, v)}{(1 - q_2)(b_2 - v)}, \quad v \neq b_2.$$

The mixed left-left partial q -derivative is defined by

$$\begin{aligned} & \frac{{}_{a_1, a_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}_{a_1}\partial_{q_1}u {}_{a_2}\partial_{q_2}v} \\ &= \frac{1}{(1 - q_1)(1 - q_2)(u - a_1)(v - a_2)} \\ & \times [\Psi(u, v) - \Psi(q_1u + (1 - q_1)a_1, v) \\ & - \Psi(u, q_2v + (1 - q_2)a_2) \\ & + \Psi(q_1u + (1 - q_1)a_1, q_2v + (1 - q_2)a_2)], \end{aligned}$$

for $u \neq a_1$ and $v \neq a_2$. The mixed left-right partial q -derivative is defined by

$$\begin{aligned} & \frac{{}^{b_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}_{a_1}\partial_{q_1}u {}^{b_2}\partial_{q_2}v} \\ &= \frac{1}{(1 - q_1)(1 - q_2)(u - a_1)(b_2 - v)} \\ & \times [\Psi(q_1u + (1 - q_1)a_1, v) + \Psi(u, q_2v + (1 - q_2)b_2) \\ & - \Psi(q_1u + (1 - q_1)a_1, q_2v + (1 - q_2)b_2) - \Psi(u, v)], \end{aligned}$$

for $u \neq a_1$ and $v \neq b_2$. The mixed right-left partial q -derivative is defined by

$$\begin{aligned} & \frac{{}_{a_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}^{b_1}\partial_{q_1}u {}_{a_2}\partial_{q_2}v} \\ &= \frac{1}{(1 - q_1)(1 - q_2)(b_1 - u)(v - a_2)} \\ & \times [\Psi(q_1u + (1 - q_1)b_1, v) + \Psi(u, q_2v + (1 - q_2)a_2) \\ & - \Psi(q_1u + (1 - q_1)b_1, q_2v + (1 - q_2)a_2) - \Psi(u, v)], \end{aligned}$$

for $u \neq b_1$ and $v \neq a_2$.

Finally, the mixed right-right partial q -derivative is defined by

$$\begin{aligned} & \frac{{}^{b_1, b_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}^{b_1}\partial_{q_1}u {}^{b_2}\partial_{q_2}v} \\ &= \frac{1}{(1 - q_1)(1 - q_2)(b_1 - u)(b_2 - v)} \\ & \times [\Psi(q_1u + (1 - q_1)b_1, q_2v + (1 - q_2)b_2) \\ & - \Psi(q_1u + (1 - q_1)b_1, v) - \Psi(u, q_2v + (1 - q_2)b_2) + \Psi(u, v)], \end{aligned}$$

for $u \neq b_1$ and $v \neq b_2$.

For convenience, the four mixed second-order partial q -derivatives used throughout this paper are denoted by

$$\varphi_1(u, v) := \frac{{}_{a_1, a_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}_{a_1}\partial_{q_1}u {}_{a_2}\partial_{q_2}v},$$

$$\varphi_2(u, v) := \frac{{}^{b_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}_{a_1}\partial_{q_1}u {}^{b_2}\partial_{q_2}v},$$

$$\varphi_3(u, v) := \frac{{}_{a_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}^{b_1}\partial_{q_1}u {}_{a_2}\partial_{q_2}v},$$

and

$$\varphi_4(u, v) := \frac{{}^{b_1, b_2}\partial_{q_1, q_2}^2 \Psi(u, v)}{{}^{b_1}\partial_{q_1}u {}^{b_2}\partial_{q_2}v}.$$

The endpoint configurations of these derivatives are, respectively,

$$(a_1, a_2), \quad (a_1, b_2), \quad (b_1, a_2), \quad (b_1, b_2).$$

Furthermore, assume that Ψ is twice continuously differentiable on an open neighborhood of Δ . Then, for every

$$(u, v) \in (a_1, b_1) \times (a_2, b_2),$$

all four mixed partial q -derivatives recover the classical mixed partial derivative as $q_1, q_2 \rightarrow 1^-$. More precisely,

$$\lim_{\substack{q_1 \rightarrow 1^- \\ q_2 \rightarrow 1^-}} \varphi_i(u, v) = \Psi_{uv}(u, v) = \frac{\partial^2 \Psi}{\partial u \partial v}(u, v), \quad i = 1, 2, 3, 4.$$

2.5. Auxiliary lemmas

We now present the auxiliary results used in the proofs of the main theorems.

Theorem 2.1 ($q_1 q_2$ -Hölder inequality for functions of two variables [35]). Let $\Delta = [a_1, b_1] \times [a_2, b_2]$, $q_1, q_2 \in (0, 1)$, and let $\Psi, \Upsilon : \Delta \rightarrow \mathbb{R}$ be q -integrable functions. If $\rho_1, \rho_2 > 1$ satisfy

$$\frac{1}{\rho_1} + \frac{1}{\rho_2} = 1,$$

then

$$\begin{aligned} & \int_{a_1}^{b_1} \int_{a_2}^{b_2} |\Psi(u, v)\Upsilon(u, v)| {}_{a_2}d_{q_2}v {}_{a_1}d_{q_1}u \\ & \leq \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} |\Psi(u, v)|^{\rho_1} {}_{a_2}d_{q_2}v {}_{a_1}d_{q_1}u \right)^{1/\rho_1} \\ & \times \left(\int_{a_1}^{b_1} \int_{a_2}^{b_2} |\Upsilon(u, v)|^{\rho_2} {}_{a_2}d_{q_2}v {}_{a_1}d_{q_1}u \right)^{1/\rho_2}. \end{aligned}$$

The following mixed quantum integral identity was established by Wannalookkhee et al. in Ref. [23].

Lemma 2.1. Let $\Phi : \Delta \rightarrow \mathbb{R}$ be a function whose four mixed partial q -derivatives $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ exist on the corresponding interiors of Δ . Suppose that these derivatives are q -integrable on Δ . Define

$$m_1 = \frac{a_1 + b_1}{2}, \quad m_2 = \frac{a_2 + b_2}{2}.$$

Then

$$\frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4}$$

$$\begin{aligned}
& + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \left(\int_{a_1}^{m_1} \int_{a_2}^{m_2} \Phi(x, y) {}_{m_2}d_{q_2}y {}_{m_1}d_{q_1}x \right. \\
& + \int_{a_1}^{m_1} \int_{m_2}^{b_2} \Phi(x, y) {}_{m_2}d_{q_2}y {}_{m_1}d_{q_1}x \\
& + \int_{m_1}^{b_1} \int_{a_2}^{m_2} \Phi(x, y) {}_{m_2}d_{q_2}y {}_{m_1}d_{q_1}x \\
& \left. + \int_{m_1}^{b_1} \int_{m_2}^{b_2} \Phi(x, y) {}_{m_2}d_{q_2}y {}_{m_1}d_{q_1}x \right) \\
& - \frac{1}{2(b_1 - a_1)} \left(\int_{a_1}^{m_1} (\Phi(x, a_2) + \Phi(x, b_2)) {}_{m_1}d_{q_1}x \right. \\
& \left. + \int_{m_1}^{b_1} (\Phi(x, a_2) + \Phi(x, b_2)) {}_{m_1}d_{q_1}x \right) \\
& - \frac{1}{2(b_2 - a_2)} \left(\int_{a_2}^{m_2} (\Phi(a_1, y) + \Phi(b_1, y)) {}_{m_2}d_{q_2}y \right. \\
& \left. + \int_{m_2}^{b_2} (\Phi(a_1, y) + \Phi(b_1, y)) {}_{m_2}d_{q_2}y \right) \\
& = \frac{(b_1 - a_1)(b_2 - a_2)}{16} \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \\
& \times \left[\varphi_1 \left(\frac{1 - \theta_1}{2} a_1 + \frac{1 + \theta_1}{2} b_1, \frac{1 - \theta_2}{2} a_2 + \frac{1 + \theta_2}{2} b_2 \right) \right. \\
& - \varphi_2 \left(\frac{1 - \theta_1}{2} a_1 + \frac{1 + \theta_1}{2} b_1, \frac{1 + \theta_2}{2} a_2 + \frac{1 - \theta_2}{2} b_2 \right) \\
& - \varphi_3 \left(\frac{1 + \theta_1}{2} a_1 + \frac{1 - \theta_1}{2} b_1, \frac{1 - \theta_2}{2} a_2 + \frac{1 + \theta_2}{2} b_2 \right) \\
& \left. + \varphi_4 \left(\frac{1 + \theta_1}{2} a_1 + \frac{1 - \theta_1}{2} b_1, \frac{1 + \theta_2}{2} a_2 + \frac{1 - \theta_2}{2} b_2 \right) \right] \\
& \times {}_0d_{q_2}\theta_2 {}_0d_{q_1}\theta_1. \tag{2.7}
\end{aligned}$$

The following estimates will be used repeatedly in the proofs.

Lemma 2.2. Let $q \in (0, 1)$ and $s \in (0, 1]$. Then

$$\begin{aligned}
\int_0^1 \theta(1 + \theta)^s {}_0d_q\theta &= (1 - q) \sum_{m=0}^{\infty} q^{2m} (1 + q^m)^s \\
&\leq \frac{1}{[2]_q} + \frac{1}{[s + 2]_q}. \tag{2.8}
\end{aligned}$$

Proof. By the definition of the Jackson q -integral,

$$\int_0^1 \theta(1 + \theta)^s {}_0d_q\theta = (1 - q) \sum_{m=0}^{\infty} q^{2m} (1 + q^m)^s.$$

Since $s \in (0, 1]$, the subadditivity of the function $x \mapsto x^s$ gives

$$(1 + \theta)^s \leq 1 + \theta^s, \quad 0 \leq \theta \leq 1.$$

Therefore,

$$\theta(1 + \theta)^s \leq \theta + \theta^{s+1}.$$

Integrating both sides yields

$$\int_0^1 \theta(1 + \theta)^s {}_0d_q\theta \leq \frac{1}{[2]_q} + \frac{1}{[s + 2]_q}.$$

□

Lemma 2.3. Let $q \in (0, 1)$ and $s \in (0, 1]$. Then

$$\begin{aligned}
\int_0^1 \theta(1 - \theta)^s {}_0d_q\theta &= (1 - q) \sum_{m=0}^{\infty} q^{2m} (1 - q^m)^s \\
&\leq \frac{1}{[2]_q} - \frac{s}{[3]_q}. \tag{2.9}
\end{aligned}$$

Proof. By the definition of the Jackson q -integral,

$$\int_0^1 \theta(1 - \theta)^s {}_0d_q\theta = (1 - q) \sum_{m=0}^{\infty} q^{2m} (1 - q^m)^s.$$

Since $x \mapsto (1 - x)^s$ is concave on $[0, 1]$ for $s \in (0, 1]$, its tangent line at $x = 0$ gives

$$(1 - \theta)^s \leq 1 - s\theta, \quad 0 \leq \theta \leq 1.$$

Hence,

$$\theta(1 - \theta)^s \leq \theta - s\theta^2.$$

It follows that

$$\begin{aligned}
\int_0^1 \theta(1 - \theta)^s {}_0d_q\theta &\leq \int_0^1 (\theta - s\theta^2) {}_0d_q\theta \\
&= \frac{1}{[2]_q} - \frac{s}{[3]_q}.
\end{aligned}$$

□

Lemma 2.4. Let $q \in (0, 1)$ and $s \in (0, 1]$. Then

$$\begin{aligned}
\int_0^1 (1 + \theta)^s {}_0d_q\theta &= (1 - q) \sum_{m=0}^{\infty} q^m (1 + q^m)^s \\
&\leq 1 + \frac{1}{[s + 1]_q}. \tag{2.10}
\end{aligned}$$

Proof. The equality follows directly from the definition of the Jackson q -integral. Since

$$(1 + \theta)^s \leq 1 + \theta^s, \quad 0 \leq \theta \leq 1,$$

we obtain

$$\begin{aligned}
\int_0^1 (1 + \theta)^s {}_0d_q\theta &\leq \int_0^1 (1 + \theta^s) {}_0d_q\theta \\
&= 1 + \frac{1}{[s + 1]_q}.
\end{aligned}$$

□

Lemma 2.5. Let $q \in (0, 1)$ and $s \in (0, 1]$. Then

$$\begin{aligned}
\int_0^1 (1 - \theta)^s {}_0d_q\theta &= (1 - q) \sum_{m=0}^{\infty} q^m (1 - q^m)^s \\
&\leq 1. \tag{2.11}
\end{aligned}$$

Proof. For $0 \leq \theta \leq 1$ and $s \in (0, 1]$,

$$0 \leq (1 - \theta)^s \leq 1.$$

Therefore,

$$\int_0^1 (1 - \theta)^s {}_0d_q\theta \leq \int_0^1 1 {}_0d_q\theta = 1.$$

□

3. Main results

In this section, we establish new q -trapezoid-type inequalities for coordinated strongly s -convex functions on bidimensional intervals. These results extend and generalize several existing inequalities by employing the identity in Lemma 2.1 together with q -integral techniques.

Theorem 3.1. Let $\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, and let $\Phi : \Delta \rightarrow \mathbb{R}$ be a function whose four mixed partial q -derivatives $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ exist on Δ° and are q -integrable on Δ . Suppose that $|\varphi_1|, |\varphi_2|, |\varphi_3|$, and $|\varphi_4|$ are coordinated strongly s -convex on Δ with respect to $(e_1, e_2), (e_3, e_4), (e_5, e_6), (e_7, e_8)$, respectively, where $e_i > 0$ for $i = 1, \dots, 8$ and $s \in (0, 1]$. Then the following inequality holds:

$$\begin{aligned} & \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\ & + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \left(\int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \right. \\ & + \int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\ & + \int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\ & + \int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \Big) \\ & - \frac{1}{2(b_1 - a_1)} \left(\int_{a_1}^{a_1+b_1} [\Phi(u, a_2) + \Phi(u, b_2)] \frac{a_1+b_1}{2} d_{q_1} u \right. \\ & + \int_{a_1}^{a_1+b_1} [\Phi(u, a_2) + \Phi(u, b_2)] \frac{a_1+b_1}{2} d_{q_1} u \Big) \\ & - \frac{1}{2(b_2 - a_2)} \left(\int_{a_2}^{a_2+b_2} [\Phi(a_1, v) + \Phi(b_1, v)] \frac{a_2+b_2}{2} d_{q_2} v \right. \\ & + \int_{a_2}^{a_2+b_2} [\Phi(a_1, v) + \Phi(b_1, v)] \frac{a_2+b_2}{2} d_{q_2} v \Big) \\ & \leq \frac{(b_1 - a_1)(b_2 - a_2)q_1q_2}{2^{2s+4}} \left[\left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) \right. \\ & \times (|\varphi_1(a_1, a_2)| + |\varphi_2(a_1, b_2)| + |\varphi_3(b_1, a_2)| + |\varphi_4(b_1, b_2)|) \\ & + \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) \\ & \times (|\varphi_1(a_1, b_2)| + |\varphi_2(a_1, a_2)| + |\varphi_3(b_1, b_2)| + |\varphi_4(b_1, a_2)|) \\ & + \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) \\ & \times (|\varphi_1(b_1, a_2)| + |\varphi_2(b_1, b_2)| + |\varphi_3(a_1, a_2)| + |\varphi_4(a_1, b_2)|) \\ & + \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) \\ & \times (|\varphi_1(b_1, b_2)| + |\varphi_2(b_1, a_2)| + |\varphi_3(a_1, b_2)| + |\varphi_4(a_1, a_2)|) \Big] \\ & - \frac{(b_1 - a_1)(b_2 - a_2)q_1^3q_2}{64[2]_{q_2}[4]_{q_1}} (e_1 + e_3 + e_5 + e_7)(b_1 - a_1)^2 \end{aligned}$$

$$- \frac{(b_1 - a_1)(b_2 - a_2)q_1q_2^3}{64[2]_{q_1}[4]_{q_2}} (e_2 + e_4 + e_6 + e_8)(b_2 - a_2)^2. \quad (3.1)$$

Proof. Applying Lemma 2.1 and then taking the absolute value, we obtain

$$\begin{aligned} & \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\ & + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \left(\int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \right. \\ & + \int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\ & + \int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\ & + \int_{a_1}^{a_1+b_1} \int_{a_2}^{a_2+b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \Big) \\ & - \frac{1}{2(b_1 - a_1)} \left(\int_{a_1}^{a_1+b_1} [\Phi(u, a_2) + \Phi(u, b_2)] \frac{a_1+b_1}{2} d_{q_1} u \right. \\ & + \int_{a_1}^{a_1+b_1} [\Phi(u, a_2) + \Phi(u, b_2)] \frac{a_1+b_1}{2} d_{q_1} u \Big) \\ & - \frac{1}{2(b_2 - a_2)} \left(\int_{a_2}^{a_2+b_2} [\Phi(a_1, v) + \Phi(b_1, v)] \frac{a_2+b_2}{2} d_{q_2} v \right. \\ & + \int_{a_2}^{a_2+b_2} [\Phi(a_1, v) + \Phi(b_1, v)] \frac{a_2+b_2}{2} d_{q_2} v \Big) \\ & \leq \frac{(b_1 - a_1)(b_2 - a_2)}{16} \int_0^1 \int_0^1 (q_1\theta_1)(q_2\theta_2) \\ & \times \left[\left| \varphi_1 \left(\frac{1-\theta_1}{2}a_1 + \frac{1+\theta_1}{2}b_1, \frac{1-\theta_2}{2}a_2 + \frac{1+\theta_2}{2}b_2 \right) \right| \right. \\ & + \left| \varphi_2 \left(\frac{1-\theta_1}{2}a_1 + \frac{1+\theta_1}{2}b_1, \frac{1+\theta_2}{2}a_2 + \frac{1-\theta_2}{2}b_2 \right) \right| \\ & + \left| \varphi_3 \left(\frac{1+\theta_1}{2}a_1 + \frac{1-\theta_1}{2}b_1, \frac{1-\theta_2}{2}a_2 + \frac{1+\theta_2}{2}b_2 \right) \right| \\ & + \left| \varphi_4 \left(\frac{1+\theta_1}{2}a_1 + \frac{1-\theta_1}{2}b_1, \frac{1+\theta_2}{2}a_2 + \frac{1-\theta_2}{2}b_2 \right) \right| \Big] \\ & {}_0d_{q_2}\theta_2 {}_0d_{q_1}\theta_1. \quad (3.2) \end{aligned}$$

By the coordinated strong s -convexity of $|\varphi_1|, |\varphi_2|, |\varphi_3|$, and $|\varphi_4|$, together with Proposition 2.1, we obtain the following estimates. For $|\varphi_1|$, we have

$$\begin{aligned} & \left| \varphi_1 \left(\frac{1-\theta_1}{2}a_1 + \frac{1+\theta_1}{2}b_1, \frac{1-\theta_2}{2}a_2 + \frac{1+\theta_2}{2}b_2 \right) \right| \\ & \leq \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_1(a_1, a_2)| \\ & + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_1(a_1, b_2)| \\ & + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_1(b_1, a_2)| \end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1+\theta_1}{2}\right)^s \left(\frac{1+\theta_2}{2}\right)^s |\varphi_1(b_1, b_2)| \\
& - e_1 \left(\frac{1-\theta_1}{2}\right) \left(\frac{1+\theta_1}{2}\right) (b_1 - a_1)^2 \\
& - e_2 \left(\frac{1-\theta_2}{2}\right) \left(\frac{1+\theta_2}{2}\right) (b_2 - a_2)^2.
\end{aligned} \quad (3.3)$$

For $|\varphi_2|$, we obtain

$$\begin{aligned}
& \left| \varphi_2 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \frac{1+\theta_2}{2} a_2 + \frac{1-\theta_2}{2} b_2 \right) \right| \\
& \leq \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_2(a_1, a_2)| \\
& + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_2(a_1, b_2)| \\
& + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_2(b_1, a_2)| \\
& + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_2(b_1, b_2)| \\
& - e_3 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (b_1 - a_1)^2 \\
& - e_4 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (b_2 - a_2)^2.
\end{aligned} \quad (3.4)$$

For $|\varphi_3|$, we have

$$\begin{aligned}
& \left| \varphi_3 \left(\frac{1+\theta_1}{2} a_1 + \frac{1-\theta_1}{2} b_1, \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right| \\
& \leq \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_3(a_1, a_2)| \\
& + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_3(a_1, b_2)| \\
& + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_3(b_1, a_2)| \\
& + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_3(b_1, b_2)| \\
& - e_5 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (b_1 - a_1)^2 \\
& - e_6 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (b_2 - a_2)^2.
\end{aligned} \quad (3.5)$$

Finally, for $|\varphi_4|$, we obtain

$$\begin{aligned}
& \left| \varphi_4 \left(\frac{1+\theta_1}{2} a_1 + \frac{1-\theta_1}{2} b_1, \frac{1+\theta_2}{2} a_2 + \frac{1-\theta_2}{2} b_2 \right) \right| \\
& \leq \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_4(a_1, a_2)| \\
& + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_4(a_1, b_2)| \\
& + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_4(b_1, a_2)| \\
& + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_4(b_1, b_2)|
\end{aligned}$$

$$\begin{aligned}
& - e_7 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (b_1 - a_1)^2 \\
& - e_8 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (b_2 - a_2)^2.
\end{aligned} \quad (3.6)$$

Since the left-hand sides of (3.3)–(3.6) are nonnegative, the corresponding right-hand sides are also nonnegative. Substituting (3.3)–(3.6) into (3.2), we obtain

$$\begin{aligned}
& \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\
& + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \left(\int_{a_1}^{\frac{a_1+b_1}{2}} \int_{a_2}^{\frac{a_2+b_2}{2}} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \right. \\
& + \int_{a_1}^{\frac{a_1+b_1}{2}} \int_{\frac{a_2+b_2}{2}}^{b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\
& + \int_{\frac{a_1+b_1}{2}}^{b_1} \int_{a_2}^{\frac{a_2+b_2}{2}} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\
& + \left. \int_{\frac{a_1+b_1}{2}}^{b_1} \int_{\frac{a_2+b_2}{2}}^{b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \right) \\
& - \frac{1}{2(b_1 - a_1)} \left(\int_{a_1}^{\frac{a_1+b_1}{2}} [\Phi(u, a_2) + \Phi(u, b_2)] \frac{a_1+b_1}{2} d_{q_1} u \right. \\
& + \left. \int_{\frac{a_1+b_1}{2}}^{b_1} [\Phi(u, a_2) + \Phi(u, b_2)] \frac{a_1+b_1}{2} d_{q_1} u \right) \\
& - \frac{1}{2(b_2 - a_2)} \left(\int_{a_2}^{\frac{a_2+b_2}{2}} [\Phi(a_1, v) + \Phi(b_1, v)] \frac{a_2+b_2}{2} d_{q_2} v \right. \\
& + \left. \int_{\frac{a_2+b_2}{2}}^{b_2} [\Phi(a_1, v) + \Phi(b_1, v)] \frac{a_2+b_2}{2} d_{q_2} v \right) \Big| \\
& \leq \frac{(b_1 - a_1)(b_2 - a_2)}{2^{2s+4}} \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \\
& \times \left[(1 - \theta_1)^s (1 - \theta_2)^s (|\varphi_1(a_1, a_2)| + |\varphi_2(a_1, b_2)| \right. \\
& + |\varphi_3(b_1, a_2)| + |\varphi_4(b_1, b_2)|) \\
& + (1 - \theta_1)^s (1 + \theta_2)^s (|\varphi_1(a_1, b_2)| + |\varphi_2(a_1, a_2)| \\
& + |\varphi_3(b_1, b_2)| + |\varphi_4(b_1, a_2)|) \\
& + (1 + \theta_1)^s (1 - \theta_2)^s (|\varphi_1(b_1, a_2)| + |\varphi_2(b_1, b_2)| \\
& + |\varphi_3(a_1, a_2)| + |\varphi_4(a_1, b_2)|) \\
& + (1 + \theta_1)^s (1 + \theta_2)^s (|\varphi_1(b_1, b_2)| + |\varphi_2(b_1, a_2)| \\
& + |\varphi_3(a_1, b_2)| + |\varphi_4(a_1, a_2)|) \Big] d_{q_2} \theta_2 d_{q_1} \theta_1 \\
& - \frac{(b_1 - a_1)(b_2 - a_2)}{64} \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \\
& \times \left[(e_1 + e_3 + e_5 + e_7)(1 - \theta_1)(1 + \theta_1)(b_1 - a_1)^2 \right. \\
& + (e_2 + e_4 + e_6 + e_8)(1 - \theta_2)(1 + \theta_2)(b_2 - a_2)^2 \Big] d_{q_2} \theta_2 d_{q_1} \theta_1.
\end{aligned} \quad (3.7)$$

It remains to estimate the six double q -integrals appearing on the right-hand side of (3.7). By Lemmas 2.2 and 2.3, we have

$$\begin{aligned} & \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2)(1 - \theta_1)^s(1 - \theta_2)^s {}_0d_{q_2} \theta_2 {}_0d_{q_1} \theta_1 \\ &= (1 - q_1)(1 - q_2)q_1 q_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^{2n} q_2^{2m} (1 - q_1^n)^s (1 - q_2^m)^s \\ &= q_1 q_2 \left[(1 - q_1) \sum_{n=0}^{\infty} q_1^{2n} (1 - q_1^n)^s \right] \\ & \quad \times \left[(1 - q_2) \sum_{m=0}^{\infty} q_2^{2m} (1 - q_2^m)^s \right] \\ &\leq q_1 q_2 \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right). \end{aligned} \quad (3.8)$$

Similarly,

$$\begin{aligned} & \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2)(1 - \theta_1)^s(1 + \theta_2)^s {}_0d_{q_2} \theta_2 {}_0d_{q_1} \theta_1 \\ &= (1 - q_1)(1 - q_2)q_1 q_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^{2n} q_2^{2m} (1 - q_1^n)^s (1 + q_2^m)^s \\ &= q_1 q_2 \left[(1 - q_1) \sum_{n=0}^{\infty} q_1^{2n} (1 - q_1^n)^s \right] \\ & \quad \times \left[(1 - q_2) \sum_{m=0}^{\infty} q_2^{2m} (1 + q_2^m)^s \right] \\ &\leq q_1 q_2 \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s + 2]_{q_2}} \right). \end{aligned} \quad (3.9)$$

Furthermore,

$$\begin{aligned} & \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2)(1 + \theta_1)^s(1 - \theta_2)^s {}_0d_{q_2} \theta_2 {}_0d_{q_1} \theta_1 \\ &= (1 - q_1)(1 - q_2)q_1 q_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^{2n} q_2^{2m} (1 + q_1^n)^s (1 - q_2^m)^s \\ &= q_1 q_2 \left[(1 - q_1) \sum_{n=0}^{\infty} q_1^{2n} (1 + q_1^n)^s \right] \\ & \quad \times \left[(1 - q_2) \sum_{m=0}^{\infty} q_2^{2m} (1 - q_2^m)^s \right] \\ &\leq q_1 q_2 \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s + 2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right). \end{aligned} \quad (3.10)$$

Finally,

$$\begin{aligned} & \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2)(1 + \theta_1)^s(1 + \theta_2)^s {}_0d_{q_2} \theta_2 {}_0d_{q_1} \theta_1 \\ &= (1 - q_1)(1 - q_2)q_1 q_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^{2n} q_2^{2m} (1 + q_1^n)^s (1 + q_2^m)^s \\ &= q_1 q_2 \left[(1 - q_1) \sum_{n=0}^{\infty} q_1^{2n} (1 + q_1^n)^s \right] \end{aligned}$$

$$\begin{aligned} & \times \left[(1 - q_2) \sum_{m=0}^{\infty} q_2^{2m} (1 + q_2^m)^s \right] \\ &\leq q_1 q_2 \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s + 2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s + 2]_{q_2}} \right). \end{aligned} \quad (3.11)$$

For the two integrals involving the strong convexity moduli, the geometric-series formula gives

$$\begin{aligned} & \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2)(1 - \theta_1)(1 + \theta_1) {}_0d_{q_2} \theta_2 {}_0d_{q_1} \theta_1 \\ &= (1 - q_1)(1 - q_2)q_1 q_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^{2n} q_2^{2m} (1 - q_1^{2n}) \\ &= q_1 q_2 \left[(1 - q_1) \sum_{n=0}^{\infty} q_1^{2n} (1 - q_1^{2n}) \right] \left[(1 - q_2) \sum_{m=0}^{\infty} q_2^{2m} \right] \\ &= q_1 q_2 \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) \frac{1}{[2]_{q_2}} \\ &= \frac{q_1^3 q_2}{[2]_{q_2} [4]_{q_1}}, \end{aligned} \quad (3.12)$$

where we have used

$$\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} = \frac{q_1^2}{[4]_{q_1}}.$$

Similarly,

$$\begin{aligned} & \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2)(1 - \theta_2)(1 + \theta_2) {}_0d_{q_2} \theta_2 {}_0d_{q_1} \theta_1 \\ &= (1 - q_1)(1 - q_2)q_1 q_2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^{2n} q_2^{2m} (1 - q_2^{2m}) \\ &= q_1 q_2 \left[(1 - q_1) \sum_{n=0}^{\infty} q_1^{2n} \right] \left[(1 - q_2) \sum_{m=0}^{\infty} q_2^{2m} (1 - q_2^{2m}) \right] \\ &= q_1 q_2 \frac{1}{[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) \\ &= \frac{q_1 q_2^3}{[2]_{q_1} [4]_{q_2}}, \end{aligned} \quad (3.13)$$

where

$$\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} = \frac{q_2^2}{[4]_{q_2}}.$$

Substituting (3.8)–(3.13) into (3.7) yields (3.1), which completes the proof. $\square$

Corollary 3.1. Let $\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, and let Φ be twice continuously differentiable on an open neighborhood of Δ . For $0 < q_1, q_2 < 1$, denote the four mixed partial q -derivatives of Φ by $\varphi_1, \varphi_2, \varphi_3$, and φ_4 .

Assume that there exists $q_0 \in (0, 1)$ such that, for all $q_1, q_2 \in (q_0, 1)$, these derivatives exist on Δ° and are q -integrable on Δ . We then consider the limit $q_1, q_2 \rightarrow 1^-$. Suppose further that $|\varphi_1|, |\varphi_2|, |\varphi_3|$, and $|\varphi_4|$ are coordinated strongly s -convex on Δ with respect to

$$(e_1, e_2), \quad (e_3, e_4), \quad (e_5, e_6), \quad (e_7, e_8),$$

respectively, where $s \in (0, 1]$ and the constants $e_i > 0$, $i = 1, \dots, 8$, are independent of q_1 and q_2 .

For convenience, write

$$\Phi_{uv}(u, v) := \frac{\partial^2 \Phi}{\partial u \partial v}(u, v).$$

Since Φ is twice continuously differentiable,

$$\lim_{\substack{q_1 \rightarrow 1^- \\ q_2 \rightarrow 1^-}} \varphi_i(u, v) = \Phi_{uv}(u, v), \quad i = 1, 2, 3, 4.$$

Therefore, letting $q_1, q_2 \rightarrow 1^-$ in Theorem 3.1, we obtain

$$\begin{aligned} & \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\ & + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Phi(u, v) dv du \\ & - \frac{1}{2(b_1 - a_1)} \int_{a_1}^{b_1} [\Phi(u, a_2) + \Phi(u, b_2)] du \\ & \left. - \frac{1}{2(b_2 - a_2)} \int_{a_2}^{b_2} [\Phi(a_1, v) + \Phi(b_1, v)] dv \right| \\ & \leq \frac{(b_1 - a_1)(b_2 - a_2)}{2^{2s+4}} \left(1 - \frac{s}{3} + \frac{1}{s+2} \right)^2 \\ & \times (|\Phi_{uv}(a_1, a_2)| + |\Phi_{uv}(a_1, b_2)| + |\Phi_{uv}(b_1, a_2)| + |\Phi_{uv}(b_1, b_2)|) \\ & - \frac{(b_1 - a_1)(b_2 - a_2)}{512} (e_1 + e_3 + e_5 + e_7)(b_1 - a_1)^2 \\ & - \frac{(b_1 - a_1)(b_2 - a_2)}{512} (e_2 + e_4 + e_6 + e_8)(b_2 - a_2)^2. \quad (3.14) \end{aligned}$$

Remark 3.1. When $s = 1$, Theorem 3.1 is analogous to Theorem 2 of Mishra *et al.* in Ref. [29]. The difference arises only from the symmetric form of the strong convexity terms adopted in the present paper.

Theorem 3.2. Let $\Phi : \Delta \rightarrow \mathbb{R}$ be a function whose four mixed partial q -derivatives $\varphi_1, \varphi_2, \varphi_3$, and φ_4 exist on Δ° . Let $i_1, i_2 > 1$ satisfy

$$\frac{1}{i_1} + \frac{1}{i_2} = 1.$$

Assume that $|\varphi_k|^{i_k}$ is q -integrable on Δ for each $k = 1, 2, 3, 4$. Suppose further that $|\varphi_1|^{i_1}$, $|\varphi_2|^{i_2}$, $|\varphi_3|^{i_3}$, and $|\varphi_4|^{i_4}$ are coordinated strongly s -convex on Δ with respect to the modulus pairs (e_1, e_2) , (e_3, e_4) , (e_5, e_6) , and (e_7, e_8) , respectively, where $e_j > 0$ for $j = 1, \dots, 8$ and $s \in (0, 1]$, then the following inequality holds:

$$\begin{aligned} & \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\ & + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \left(\int_{a_1}^{a_1 + \frac{b_1 - a_1}{2}} \int_{a_2}^{a_2 + \frac{b_2 - a_2}{2}} \Phi(u, v) \frac{a_2 + b_2}{2} d_{q_2} v \frac{a_1 + b_1}{2} d_{q_1} u \right. \\ & \left. + \int_{a_1}^{a_1 + \frac{b_1 - a_1}{2}} \int_{a_2}^{a_2 + \frac{b_2 - a_2}{2}} \Phi(u, v) \frac{a_2 + b_2}{2} d_{q_2} v \frac{a_1 + b_1}{2} d_{q_1} u \right. \end{aligned}$$

$$\begin{aligned} & + \int_{a_1 + \frac{b_1 - a_1}{2}}^{b_1} \int_{a_2}^{a_2 + \frac{b_2 - a_2}{2}} \Phi(u, v) \frac{a_2 + b_2}{2} d_{q_2} v \frac{a_1 + b_1}{2} d_{q_1} u \\ & + \int_{a_1 + \frac{b_1 - a_1}{2}}^{b_1} \int_{a_2}^{a_2 + \frac{b_2 - a_2}{2}} \Phi(u, v) \frac{a_2 + b_2}{2} d_{q_2} v \frac{a_1 + b_1}{2} d_{q_1} u \Big) \\ & - \frac{1}{2(b_1 - a_1)} \left(\int_{a_1}^{a_1 + \frac{b_1 - a_1}{2}} (\Phi(u, a_2) + \Phi(u, b_2)) \frac{a_1 + b_1}{2} d_{q_1} u \right. \\ & + \int_{a_1 + \frac{b_1 - a_1}{2}}^{b_1} (\Phi(u, a_2) + \Phi(u, b_2)) \frac{a_1 + b_1}{2} d_{q_1} u \Big) \\ & - \frac{1}{2(b_2 - a_2)} \left(\int_{a_2}^{a_2 + \frac{b_2 - a_2}{2}} (\Phi(a_1, v) + \Phi(b_1, v)) \frac{a_2 + b_2}{2} d_{q_2} v \right. \\ & + \int_{a_2 + \frac{b_2 - a_2}{2}}^{b_2} (\Phi(a_1, v) + \Phi(b_1, v)) \frac{a_2 + b_2}{2} d_{q_2} v \Big) \Big| \\ & \leq \frac{(b_1 - a_1)(b_2 - a_2)}{16} \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2 + 1]_{q_1} [i_2 + 1]_{q_2}} \right)^{\frac{1}{i_2}} \\ & \times \left[\left(\frac{1}{2^{2s}} |\varphi_1(a_1, a_2)|^{i_1} + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_1(a_1, b_2)|^{i_1} \right. \right. \\ & + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) |\varphi_1(b_1, a_2)|^{i_1} \\ & + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_1(b_1, b_2)|^{i_1} \\ & - e_1 \left(\frac{q_1 [2]_{q_1}}{4 [3]_{q_1}} \right) (a_1 - b_1)^2 - e_2 \left(\frac{q_2 [2]_{q_2}}{4 [3]_{q_2}} \right) (a_2 - b_2)^2 \Big)^{\frac{1}{i_1}} \\ & + \left(\frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_2(a_1, a_2)|^{i_1} \right. \\ & + \frac{1}{2^{2s}} |\varphi_2(a_1, b_2)|^{i_1} \\ & + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_2(b_1, a_2)|^{i_1} \\ & + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) |\varphi_2(b_1, b_2)|^{i_1} \\ & - e_3 \left(\frac{q_1 [2]_{q_1}}{4 [3]_{q_1}} \right) (a_1 - b_1)^2 - e_4 \left(\frac{q_2 [2]_{q_2}}{4 [3]_{q_2}} \right) (a_2 - b_2)^2 \Big)^{\frac{1}{i_1}} \\ & + \left(\frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) |\varphi_3(a_1, a_2)|^{i_1} \right. \\ & + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_3(a_1, b_2)|^{i_1} \\ & + \frac{1}{2^{2s}} |\varphi_3(b_1, a_2)|^{i_1} + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_3(b_1, b_2)|^{i_1} \\ & - e_5 \left(\frac{q_1 [2]_{q_1}}{4 [3]_{q_1}} \right) (a_1 - b_1)^2 - e_6 \left(\frac{q_2 [2]_{q_2}}{4 [3]_{q_2}} \right) (a_2 - b_2)^2 \Big)^{\frac{1}{i_1}} \\ & + \left(\frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_4(a_1, a_2)|^{i_1} \right. \\ & + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) |\varphi_4(a_1, b_2)|^{i_1} \end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_2}} \right) |\varphi_4(b_1, a_2)|^{i_1} \\
& + \frac{1}{2^{2s}} |\varphi_4(b_1, b_2)|^{i_1} - e_7 \left(\frac{q_1[2]_{q_1}}{4[3]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_8 \left(\frac{q_2[2]_{q_2}}{4[3]_{q_2}} \right) (a_2 - b_2)^2 \Big)^{\frac{1}{i_1}}. \tag{3.15}
\end{aligned}$$

Proof. To apply the q_1q_2 -Hölder inequality, we first evaluate

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1\theta_1)^{i_2} (q_2\theta_2)^{i_2} d_{q_2}\theta_2 d_{q_1}\theta_1 \\
& = q_1^{i_2} q_2^{i_2} \left(\int_0^1 \theta_1^{i_2} d_{q_1}\theta_1 \right) \left(\int_0^1 \theta_2^{i_2} d_{q_2}\theta_2 \right) \\
& = \frac{q_1^{i_2} q_2^{i_2}}{[i_2+1]_{q_1} [i_2+1]_{q_2}}. \tag{3.16}
\end{aligned}$$

Applying the q_1q_2 -Hölder inequality together with (3.16), and then using the coordinated strongly s -convexity of $|\varphi_k(u, v)|^{i_1}$, for $k = 1, 2, 3, 4$, as well as Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1\theta_1)(q_2\theta_2) \left| \varphi_1 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \right. \right. \\
& \quad \left. \left. \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right| d_{q_2}\theta_2 d_{q_1}\theta_1 \\
& \leq \left(\int_0^1 \int_0^1 (q_1\theta_1)^{i_2} (q_2\theta_2)^{i_2} d_{q_2}\theta_2 d_{q_1}\theta_1 \right)^{\frac{1}{i_2}} \\
& \quad \times \left(\int_0^1 \int_0^1 \left| \varphi_1 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \right. \right. \right. \\
& \quad \left. \left. \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right|^{i_1} d_{q_2}\theta_2 d_{q_1}\theta_1 \Big)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2+1]_{q_1} [i_2+1]_{q_2}} \right)^{\frac{1}{i_2}} \\
& \quad \times \left(\int_0^1 \int_0^1 \left[\left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_1(a_1, a_2)|^{i_1} \right. \right. \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_1(a_1, b_2)|^{i_1} \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_1(b_1, a_2)|^{i_1} \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_1(b_1, b_2)|^{i_1} \\
& \quad - e_1 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (a_1 - b_1)^2 \\
& \quad \left. \left. - e_2 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (a_2 - b_2)^2 \right] d_{q_2}\theta_2 d_{q_1}\theta_1 \right)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2+1]_{q_1} [i_2+1]_{q_2}} \right)^{\frac{1}{i_2}} \left(\frac{1}{2^{2s}} |\varphi_1(a_1, a_2)|^{i_1} \right. \\
& \quad + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_2}} \right) |\varphi_1(a_1, b_2)|^{i_1} \\
& \quad \left. + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_1}} \right) |\varphi_1(b_1, a_2)|^{i_1} \right.
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_1}} \right) \left(1 + \frac{1}{[s+1]_{q_2}} \right) |\varphi_1(b_1, b_2)|^{i_1} \\
& - e_1 \left(\frac{q_1[2]_{q_1}}{4[3]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_2 \left(\frac{q_2[2]_{q_2}}{4[3]_{q_2}} \right) (a_2 - b_2)^2 \Big)^{\frac{1}{i_1}}. \tag{3.17}
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1\theta_1)(q_2\theta_2) \left| \varphi_2 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \right. \right. \\
& \quad \left. \left. \frac{1+\theta_2}{2} a_2 + \frac{1-\theta_2}{2} b_2 \right) \right| d_{q_2}\theta_2 d_{q_1}\theta_1 \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2+1]_{q_1} [i_2+1]_{q_2}} \right)^{\frac{1}{i_2}} \\
& \quad \times \left(\int_0^1 \int_0^1 \left[\left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_2(a_1, a_2)|^{i_1} \right. \right. \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_2(a_1, b_2)|^{i_1} \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_2(b_1, a_2)|^{i_1} \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_2(b_1, b_2)|^{i_1} \\
& \quad - e_3 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (a_1 - b_1)^2 \\
& \quad \left. \left. - e_4 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (a_2 - b_2)^2 \right] d_{q_2}\theta_2 d_{q_1}\theta_1 \right)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2+1]_{q_1} [i_2+1]_{q_2}} \right)^{\frac{1}{i_2}} \left(\frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_2}} \right) |\varphi_2(a_1, a_2)|^{i_1} \right. \\
& \quad + \frac{1}{2^{2s}} |\varphi_2(a_1, b_2)|^{i_1} \\
& \quad + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_1}} \right) \left(1 + \frac{1}{[s+1]_{q_2}} \right) |\varphi_2(b_1, a_2)|^{i_1} \\
& \quad + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s+1]_{q_1}} \right) |\varphi_2(b_1, b_2)|^{i_1} \\
& \quad - e_3 \left(\frac{q_1[2]_{q_1}}{4[3]_{q_1}} \right) (a_1 - b_1)^2 \\
& \quad \left. \left. - e_4 \left(\frac{q_2[2]_{q_2}}{4[3]_{q_2}} \right) (a_2 - b_2)^2 \right)^{\frac{1}{i_1}}, \tag{3.18}
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1\theta_1)(q_2\theta_2) \left| \varphi_3 \left(\frac{1+\theta_1}{2} a_1 + \frac{1-\theta_1}{2} b_1, \right. \right. \\
& \quad \left. \left. \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right| d_{q_2}\theta_2 d_{q_1}\theta_1 \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2+1]_{q_1} [i_2+1]_{q_2}} \right)^{\frac{1}{i_2}} \\
& \quad \times \left(\int_0^1 \int_0^1 \left[\left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_3(a_1, a_2)|^{i_1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& + \left(\frac{1+\theta_1}{2}\right)^s \left(\frac{1+\theta_2}{2}\right)^s |\varphi_3(a_1, b_2)|^{i_1} \\
& + \left(\frac{1-\theta_1}{2}\right)^s \left(\frac{1-\theta_2}{2}\right)^s |\varphi_3(b_1, a_2)|^{i_1} \\
& + \left(\frac{1-\theta_1}{2}\right)^s \left(\frac{1+\theta_2}{2}\right)^s |\varphi_3(b_1, b_2)|^{i_1} \\
& - e_5 \left(\frac{1-\theta_1}{2}\right) \left(\frac{1+\theta_1}{2}\right) (a_1 - b_1)^2 \\
& - e_6 \left(\frac{1-\theta_2}{2}\right) \left(\frac{1+\theta_2}{2}\right) (a_2 - b_2)^2 \Big] {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \Big)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2 + 1]_{q_1} [i_2 + 1]_{q_2}} \right)^{\frac{1}{i_2}} \left(\frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) |\varphi_3(a_1, a_2)|^{i_1} \right. \\
& + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_3(a_1, b_2)|^{i_1} \\
& + \frac{1}{2^{2s}} |\varphi_3(b_1, a_2)|^{i_1} \\
& + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_3(b_1, b_2)|^{i_1} \\
& - e_5 \left(\frac{q_1 [2]_{q_1}}{4 [3]_{q_1}} \right) (a_1 - b_1)^2 \\
& \left. - e_6 \left(\frac{q_2 [2]_{q_2}}{4 [3]_{q_2}} \right) (a_2 - b_2)^2 \right)^{\frac{1}{i_1}}, \tag{3.19}
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left| \varphi_4 \left(\frac{1+\theta_1}{2} a_1 + \frac{1-\theta_1}{2} b_1, \right. \right. \\
& \quad \left. \left. \frac{1+\theta_2}{2} a_2 + \frac{1-\theta_2}{2} b_2 \right) \right| {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2 + 1]_{q_1} [i_2 + 1]_{q_2}} \right)^{\frac{1}{i_2}} \\
& \quad \times \left(\int_0^1 \int_0^1 \left[\left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_4(a_1, a_2)|^{i_1} \right. \right. \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_4(a_1, b_2)|^{i_1} \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_4(b_1, a_2)|^{i_1} \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_4(b_1, b_2)|^{i_1} \\
& \quad - e_7 \left(\frac{1+\theta_1}{2} \right) \left(\frac{1-\theta_1}{2} \right) (a_1 - b_1)^2 \\
& \quad \left. \left. - e_8 \left(\frac{1+\theta_2}{2} \right) \left(\frac{1-\theta_2}{2} \right) (a_2 - b_2)^2 \right] {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \right)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2 + 1]_{q_1} [i_2 + 1]_{q_2}} \right)^{\frac{1}{i_2}} \\
& \quad \times \left(\frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_4(a_1, a_2)|^{i_1} \right. \\
& \quad + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_1}} \right) |\varphi_4(a_1, b_2)|^{i_1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{1}{2^{2s}} \left(1 + \frac{1}{[s + 1]_{q_2}} \right) |\varphi_4(b_1, a_2)|^{i_1} \\
& + \frac{1}{2^{2s}} |\varphi_4(b_1, b_2)|^{i_1} - e_7 \left(\frac{q_1 [2]_{q_1}}{4 [3]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_8 \left(\frac{q_2 [2]_{q_2}}{4 [3]_{q_2}} \right) (a_2 - b_2)^2 \Big)^{\frac{1}{i_1}}. \tag{3.20}
\end{aligned}$$

Combining (3.17)–(3.20) with (3.2), we obtain (3.15). This completes the proof. $\square$

Corollary 3.2. Let $\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, and let Φ be twice continuously differentiable on an open neighborhood of Δ . Let $i_1, i_2 > 1$ satisfy

$$\frac{1}{i_1} + \frac{1}{i_2} = 1.$$

For $0 < q_1, q_2 < 1$, denote the four mixed partial q -derivatives of Φ by $\varphi_1, \varphi_2, \varphi_3$, and φ_4 . Assume that there exists $q_0 \in (0, 1)$ such that, for all $q_1, q_2 \in (q_0, 1)$, these derivatives exist on Δ° and are q -integrable on Δ . Suppose further that

$$|\varphi_1|^{i_1}, \quad |\varphi_2|^{i_1}, \quad |\varphi_3|^{i_1}, \quad |\varphi_4|^{i_1}$$

are coordinated strongly s -convex on Δ with respect to

$$(e_1, e_2), \quad (e_3, e_4), \quad (e_5, e_6), \quad (e_7, e_8),$$

respectively, where $s \in (0, 1]$ and the constants $e_j > 0$, $j = 1, \dots, 8$, are independent of q_1 and q_2 .

For convenience, write

$$\Phi_{uv}(u, v) := \frac{\partial^2 \Phi}{\partial u \partial v}(u, v).$$

Then

$$\lim_{\substack{q_1 \rightarrow 1^- \\ q_2 \rightarrow 1^-}} |\varphi_k(u, v)|^{i_1} = |\Phi_{uv}(u, v)|^{i_1}, \quad k = 1, 2, 3, 4.$$

Therefore, letting $q_1, q_2 \rightarrow 1^-$ in Theorem 3.2, we obtain

$$\begin{aligned}
& \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\
& \quad + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Phi(u, v) dv du \\
& \quad - \frac{1}{2(b_1 - a_1)} \int_{a_1}^{b_1} [\Phi(u, a_2) + \Phi(u, b_2)] du \\
& \quad \left. - \frac{1}{2(b_2 - a_2)} \int_{a_2}^{b_2} [\Phi(a_1, v) + \Phi(b_1, v)] dv \right| \\
& \leq \frac{(b_1 - a_1)(b_2 - a_2)}{16(i_2 + 1)^{2/i_2}} \left[\left(\frac{|\Phi_{uv}(a_1, a_2)|^{i_1}}{2^{2s}} \right. \right. \\
& \quad + \frac{s + 2}{2^{2s}(s + 1)} |\Phi_{uv}(a_1, b_2)|^{i_1} + \frac{s + 2}{2^{2s}(s + 1)} |\Phi_{uv}(b_1, a_2)|^{i_1} \\
& \quad \left. \left. + \frac{(s + 2)^2}{2^{2s}(s + 1)^2} |\Phi_{uv}(b_1, b_2)|^{i_1} \right) \right]
\end{aligned}$$

$$\begin{aligned}
& -\frac{e_1}{6}(b_1 - a_1)^2 - \frac{e_2}{6}(b_2 - a_2)^2 \Big)^{1/i_1} \\
& + \left(\frac{s+2}{2^{2s}(s+1)} |\Phi_{uv}(a_1, a_2)|^{i_1} + \frac{|\Phi_{uv}(a_1, b_2)|^{i_1}}{2^{2s}} \right. \\
& + \frac{(s+2)^2}{2^{2s}(s+1)^2} |\Phi_{uv}(b_1, a_2)|^{i_1} + \frac{s+2}{2^{2s}(s+1)} |\Phi_{uv}(b_1, b_2)|^{i_1} \\
& - \frac{e_3}{6}(b_1 - a_1)^2 - \frac{e_4}{6}(b_2 - a_2)^2 \Big)^{1/i_1} \\
& + \left(\frac{s+2}{2^{2s}(s+1)} |\Phi_{uv}(a_1, a_2)|^{i_1} \right. \\
& + \frac{(s+2)^2}{2^{2s}(s+1)^2} |\Phi_{uv}(a_1, b_2)|^{i_1} + \frac{|\Phi_{uv}(b_1, a_2)|^{i_1}}{2^{2s}} \\
& + \frac{s+2}{2^{2s}(s+1)} |\Phi_{uv}(b_1, b_2)|^{i_1} \\
& - \frac{e_5}{6}(b_1 - a_1)^2 - \frac{e_6}{6}(b_2 - a_2)^2 \Big)^{1/i_1} \\
& + \left(\frac{(s+2)^2}{2^{2s}(s+1)^2} |\Phi_{uv}(a_1, a_2)|^{i_1} \right. \\
& + \frac{s+2}{2^{2s}(s+1)} |\Phi_{uv}(a_1, b_2)|^{i_1} + \frac{s+2}{2^{2s}(s+1)} |\Phi_{uv}(b_1, a_2)|^{i_1} \\
& + \frac{|\Phi_{uv}(b_1, b_2)|^{i_1}}{2^{2s}} \\
& \left. - \frac{e_7}{6}(b_1 - a_1)^2 - \frac{e_8}{6}(b_2 - a_2)^2 \Big)^{1/i_1} \right]. \quad (3.21)
\end{aligned}$$

Theorem 3.3. Let $\Phi : \Delta \rightarrow \mathbb{R}$ be a function whose four mixed partial q -derivatives $\varphi_1, \varphi_2, \varphi_3$, and φ_4 exist on Δ° . Let $i_1 \geq 1$. Assume that $|\varphi_k|^{i_1}$ is q -integrable on Δ for each $k = 1, 2, 3, 4$. Suppose further that $|\varphi_1|^{i_1}, |\varphi_2|^{i_1}, |\varphi_3|^{i_1}$, and $|\varphi_4|^{i_1}$ are coordinated strongly s -convex on Δ with respect to the modulus pairs $(e_1, e_2), (e_3, e_4), (e_5, e_6)$, and (e_7, e_8) , respectively, where $e_j > 0$ for $j = 1, \dots, 8$ and $s \in (0, 1]$. Then the following inequality holds:

$$\begin{aligned}
& \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\
& + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \left(\int_{a_1}^{\frac{a_1+b_1}{2}} \int_{a_2}^{\frac{a_2+b_2}{2}} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \right. \\
& + \int_{a_1}^{\frac{a_1+b_1}{2}} \int_{\frac{a_2+b_2}{2}}^{b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\
& + \int_{\frac{a_1+b_1}{2}}^{b_1} \int_{a_2}^{\frac{a_2+b_2}{2}} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \\
& \left. + \int_{\frac{a_1+b_1}{2}}^{b_1} \int_{\frac{a_2+b_2}{2}}^{b_2} \Phi(u, v) \frac{a_2+b_2}{2} d_{q_2} v \frac{a_1+b_1}{2} d_{q_1} u \right) \\
& - \frac{1}{2(b_1 - a_1)} \left(\int_{a_1}^{\frac{a_1+b_1}{2}} (\Phi(u, a_2) + \Phi(u, b_2)) \frac{a_1+b_1}{2} d_{q_1} u \right. \\
& \left. + \int_{\frac{a_1+b_1}{2}}^{b_1} (\Phi(u, a_2) + \Phi(u, b_2)) \frac{a_1+b_1}{2} d_{q_1} u \right)
\end{aligned}$$

$$\begin{aligned}
& - \frac{1}{2(b_2 - a_2)} \left(\int_{a_2}^{\frac{a_2+b_2}{2}} (\Phi(a_1, v) + \Phi(b_1, v)) \frac{a_2+b_2}{2} d_{q_2} v \right. \\
& \left. + \int_{\frac{a_2+b_2}{2}}^{b_2} (\Phi(a_1, v) + \Phi(b_1, v)) \frac{a_2+b_2}{2} d_{q_2} v \right) \Big| \\
& \leq \frac{(b_1 - a_1)(b_2 - a_2)}{16} \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \times \left\{ \left[\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_1(a_1, a_2)|^{i_1} \right. \right. \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_1(a_1, b_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_1(b_1, a_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_1(b_1, b_2)|^{i_1} \\
& - e_1 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_2 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \Big]^{\frac{1}{i_1}} \\
& + \left[\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_2(a_1, a_2)|^{i_1} \right. \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_2(a_1, b_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_2(b_1, a_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_2(b_1, b_2)|^{i_1} \\
& - e_3 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_4 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \Big]^{\frac{1}{i_1}} \\
& + \left[\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_3(a_1, a_2)|^{i_1} \right. \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_3(a_1, b_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_3(b_1, a_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_3(b_1, b_2)|^{i_1} \\
& - e_5 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_6 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \Big]^{\frac{1}{i_1}} \\
& + \left[\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_4(a_1, a_2)|^{i_1} \right. \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_4(a_1, b_2)|^{i_1}
\end{aligned}$$

$$\begin{aligned}
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_4(b_1, a_2)|^{i_1} \\
& + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_4(b_1, b_2)|^{i_1} \\
& - e_7 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \\
& - e_8 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \Bigg]^{i_1} \Bigg\}. \quad (3.22)
\end{aligned}$$

Proof. Using the power mean inequality for double q -integrals, we first note that

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \\
& = \frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}}. \quad (3.23)
\end{aligned}$$

Moreover, by Lemmas 2.2 and 2.3, we have

$$\begin{aligned}
& \int_0^1 q_j \theta \left(\frac{1-\theta}{2} \right)^s {}_0 d_{q_j} \theta \\
& \leq \frac{q_j}{2^s} \left(\frac{1}{[2]_{q_j}} - \frac{s}{[3]_{q_j}} \right), \\
& \int_0^1 q_j \theta \left(\frac{1+\theta}{2} \right)^s {}_0 d_{q_j} \theta \\
& \leq \frac{q_j}{2^s} \left(\frac{1}{[2]_{q_j}} + \frac{1}{[s+2]_{q_j}} \right), \quad j = 1, 2. \quad (3.24)
\end{aligned}$$

Furthermore,

$$\begin{aligned}
& \int_0^1 q_j \theta \left(\frac{1-\theta}{2} \right) \left(\frac{1+\theta}{2} \right) {}_0 d_{q_j} \theta \\
& = \frac{q_j}{4} \left(\frac{1}{[2]_{q_j}} - \frac{1}{[4]_{q_j}} \right), \quad j = 1, 2. \quad (3.25)
\end{aligned}$$

Applying the power mean inequality together with the co-ordinated strongly s -convexity of $|\varphi_1(u, v)|^{i_1}$, we obtain

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left| \varphi_1 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \right. \right. \\
& \left. \left. \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right| {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \\
& \leq \left(\int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left| \varphi_1 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \right. \right. \right. \\
& \quad \left. \left. \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right|^{i_1} {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \Bigg)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left[\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_1(a_1, a_2)|^{i_1} \right. \\
& \quad \left. + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_1(a_1, b_2)|^{i_1} \right. \\
& \quad \left. + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_1(b_1, a_2)|^{i_1} \right. \\
& \quad \left. + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_1(b_1, b_2)|^{i_1} \right. \\
& \quad \left. - e_3 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \right. \\
& \quad \left. - e_4 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \right]^{i_1}, \quad (3.26)
\end{aligned}$$

Similarly, we have

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left| \varphi_2 \left(\frac{1-\theta_1}{2} a_1 + \frac{1+\theta_1}{2} b_1, \right. \right. \\
& \left. \left. \frac{1+\theta_2}{2} a_2 + \frac{1-\theta_2}{2} b_2 \right) \right| {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \\
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left[\left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_2(a_1, a_2)|^{i_1} \right. \right. \\
& \quad \left. \left. + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_2(a_1, b_2)|^{i_1} \right. \right. \\
& \quad \left. \left. + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_2(b_1, a_2)|^{i_1} \right. \right. \\
& \quad \left. \left. + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_2(b_1, b_2)|^{i_1} \right. \right. \\
& \quad \left. \left. - e_3 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (a_1 - b_1)^2 \right. \right. \\
& \quad \left. \left. - e_4 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (a_2 - b_2)^2 \right] {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \Bigg)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_2(a_1, a_2)|^{i_1} \right. \\
& \quad \left. + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_2(a_1, b_2)|^{i_1} \right. \\
& \quad \left. + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_2(b_1, a_2)|^{i_1} \right. \\
& \quad \left. + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_2(b_1, b_2)|^{i_1} \right. \\
& \quad \left. - e_3 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \right. \\
& \quad \left. - e_4 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \right]^{i_1}, \quad (3.27)
\end{aligned}$$

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left| \varphi_3 \left(\frac{1+\theta_1}{2} a_1 + \frac{1-\theta_1}{2} b_1, \right. \right. \\
& \left. \left. \frac{1-\theta_2}{2} a_2 + \frac{1+\theta_2}{2} b_2 \right) \right| {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1
\end{aligned}$$

$$\begin{aligned}
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left[\left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_3(a_1, a_2)|^{i_1} \right. \right. \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_3(a_1, b_2)|^{i_1} \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_3(b_1, a_2)|^{i_1} \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_3(b_1, b_2)|^{i_1} \\
& \quad - e_5 \left(\frac{1-\theta_1}{2} \right) \left(\frac{1+\theta_1}{2} \right) (a_1 - b_1)^2 \\
& \quad \left. - e_6 \left(\frac{1-\theta_2}{2} \right) \left(\frac{1+\theta_2}{2} \right) (a_2 - b_2)^2 \right] {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \Big)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_3(a_1, a_2)|^{i_1} \right. \\
& \quad + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_3(a_1, b_2)|^{i_1} \\
& \quad + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_3(b_1, a_2)|^{i_1} \\
& \quad + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_3(b_1, b_2)|^{i_1} \\
& \quad - e_5 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \\
& \quad \left. - e_6 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \right)^{\frac{1}{i_1}}, \quad (3.28)
\end{aligned}$$

and

$$\begin{aligned}
& \int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left| \varphi_4 \left(\frac{1+\theta_1}{2} a_1 + \frac{1-\theta_1}{2} b_1, \right. \right. \\
& \quad \left. \left. \frac{1+\theta_2}{2} a_2 + \frac{1-\theta_2}{2} b_2 \right) \right| {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \\
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\int_0^1 \int_0^1 (q_1 \theta_1)(q_2 \theta_2) \left[\left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_4(a_1, a_2)|^{i_1} \right. \right. \\
& \quad + \left(\frac{1+\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_4(a_1, b_2)|^{i_1} \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1+\theta_2}{2} \right)^s |\varphi_4(b_1, a_2)|^{i_1} \\
& \quad + \left(\frac{1-\theta_1}{2} \right)^s \left(\frac{1-\theta_2}{2} \right)^s |\varphi_4(b_1, b_2)|^{i_1} \\
& \quad \left. - e_7 \left(\frac{1+\theta_1}{2} \right) \left(\frac{1-\theta_1}{2} \right) (a_1 - b_1)^2 \right.
\end{aligned}$$

$$\begin{aligned}
& \left. - e_8 \left(\frac{1+\theta_2}{2} \right) \left(\frac{1-\theta_2}{2} \right) (a_2 - b_2)^2 \right] {}_0 d_{q_2} \theta_2 {}_0 d_{q_1} \theta_1 \Big)^{\frac{1}{i_1}} \\
& \leq \left(\frac{q_1 q_2}{[2]_{q_1} [2]_{q_2}} \right)^{1-\frac{1}{i_1}} \\
& \quad \times \left(\frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_4(a_1, a_2)|^{i_1} \right. \\
& \quad + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_4(a_1, b_2)|^{i_1} \\
& \quad + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) |\varphi_4(b_1, a_2)|^{i_1} \\
& \quad + \frac{q_1 q_2}{2^{2s}} \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) |\varphi_4(b_1, b_2)|^{i_1} \\
& \quad - e_7 \frac{q_1 q_2}{4[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) (a_1 - b_1)^2 \\
& \quad \left. - e_8 \frac{q_1 q_2}{4[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right) (a_2 - b_2)^2 \right)^{\frac{1}{i_1}}. \quad (3.29)
\end{aligned}$$

Combining (3.26)–(3.29) with (3.2), and collecting the four resulting terms, we obtain (3.22). This completes the proof. $\square$

Corollary 3.3. Let $\Delta = [a_1, b_1] \times [a_2, b_2] \subseteq \mathbb{R}^2$, and let Φ be twice continuously differentiable on an open neighborhood of Δ . Let $i_1 \geq 1$ and suppose that

$$|\Phi_{uv}|^{i_1} = \left| \frac{\partial^2 \Phi}{\partial u \partial v} \right|^{i_1}$$

is coordinated strongly s -convex on Δ with respect to each of the modulus pairs

$$(e_1, e_2), \quad (e_3, e_4), \quad (e_5, e_6), \quad (e_7, e_8),$$

where $e_j > 0$ for $j = 1, \dots, 8$ and $s \in (0, 1]$. Set

$$A_s = \frac{1}{2} - \frac{s}{3}, \quad B_s = \frac{1}{2} + \frac{1}{s+2}.$$

Then, by taking the limits $q_1, q_2 \rightarrow 1^-$ in Theorem 3.3, we obtain

$$\begin{aligned}
& \left| \frac{\Phi(a_1, a_2) + \Phi(a_1, b_2) + \Phi(b_1, a_2) + \Phi(b_1, b_2)}{4} \right. \\
& \quad + \frac{1}{(b_1 - a_1)(b_2 - a_2)} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \Phi(u, v) dv du \\
& \quad - \frac{1}{2(b_1 - a_1)} \int_{a_1}^{b_1} [\Phi(u, a_2) + \Phi(u, b_2)] du \\
& \quad \left. - \frac{1}{2(b_2 - a_2)} \int_{a_2}^{b_2} [\Phi(a_1, v) + \Phi(b_1, v)] dv \right| \\
& \leq \frac{(b_1 - a_1)(b_2 - a_2)}{16} \left(\frac{1}{4} \right)^{1-\frac{1}{i_1}} \left\{ \left[\frac{1}{2^{2s}} (A_s^2 |\Phi_{uv}(a_1, a_2)|^{i_1} \right. \right. \right. \\
& \quad + A_s B_s |\Phi_{uv}(a_1, b_2)|^{i_1} + A_s B_s |\Phi_{uv}(b_1, a_2)|^{i_1} \\
& \quad \left. \left. + B_s^2 |\Phi_{uv}(b_1, b_2)|^{i_1} \right) \right\}
\end{aligned}$$

$$\begin{aligned}
& -\frac{e_1}{32}(b_1 - a_1)^2 - \frac{e_2}{32}(b_2 - a_2)^2 \Bigg]^{1/i_1} \\
& + \left[\frac{1}{2^{2s}} (A_s B_s |\Phi_{uv}(a_1, a_2)|^{i_1} + A_s^2 |\Phi_{uv}(a_1, b_2)|^{i_1} \right. \\
& + B_s^2 |\Phi_{uv}(b_1, a_2)|^{i_1} + A_s B_s |\Phi_{uv}(b_1, b_2)|^{i_1}) \\
& - \frac{e_3}{32}(b_1 - a_1)^2 - \frac{e_4}{32}(b_2 - a_2)^2 \Bigg]^{1/i_1} \\
& + \left[\frac{1}{2^{2s}} (A_s B_s |\Phi_{uv}(a_1, a_2)|^{i_1} + B_s^2 |\Phi_{uv}(a_1, b_2)|^{i_1} \right. \\
& + A_s^2 |\Phi_{uv}(b_1, a_2)|^{i_1} + A_s B_s |\Phi_{uv}(b_1, b_2)|^{i_1}) \\
& - \frac{e_5}{32}(b_1 - a_1)^2 - \frac{e_6}{32}(b_2 - a_2)^2 \Bigg]^{1/i_1} \\
& + \left[\frac{1}{2^{2s}} (B_s^2 |\Phi_{uv}(a_1, a_2)|^{i_1} + A_s B_s |\Phi_{uv}(a_1, b_2)|^{i_1} \right. \\
& + A_s B_s |\Phi_{uv}(b_1, a_2)|^{i_1} + A_s^2 |\Phi_{uv}(b_1, b_2)|^{i_1}) \\
& - \frac{e_7}{32}(b_1 - a_1)^2 - \frac{e_8}{32}(b_2 - a_2)^2 \Bigg]^{1/i_1} \Bigg\}. \quad (3.30)
\end{aligned}$$

4. Numerical examples

In this section, we present several numerical examples to illustrate the obtained results and verify the established inequalities.

Example 4.1. Consider the function

$$\Psi(u, v) = u^2 v^2, \quad (u, v) \in [1, 3] \times [1, 3].$$

By Definitions 2.13 and 2.14, we obtain

$$\begin{aligned}
\varphi_1(u, v) &= \frac{{}_1 I_{q_1, q_2}^2 \Psi(u, v)}{{}_1 \partial_{q_1} u {}_1 \partial_{q_2} v} \\
&= ([2]_{q_1} u + 1 - q_1)([2]_{q_2} v + 1 - q_2), \\
\varphi_2(u, v) &= \frac{{}_1^3 \partial_{q_1, q_2}^2 \Psi(u, v)}{{}_1 \partial_{q_1} u {}_3 \partial_{q_2} v} \\
&= ([2]_{q_1} u + 1 - q_1)([2]_{q_2} v + 3(1 - q_2)), \\
\varphi_3(u, v) &= \frac{{}_1^3 \partial_{q_1, q_2}^2 \Psi(u, v)}{{}_3 \partial_{q_1} u {}_1 \partial_{q_2} v} \\
&= ([2]_{q_1} u + 3(1 - q_1))([2]_{q_2} v + 1 - q_2),
\end{aligned}$$

and

$$\begin{aligned}
\varphi_4(u, v) &= \frac{{}_3^3 \partial_{q_1, q_2}^2 \Psi(u, v)}{{}_3 \partial_{q_1} u {}_3 \partial_{q_2} v} \\
&= ([2]_{q_1} u + 3(1 - q_1))([2]_{q_2} v + 3(1 - q_2)).
\end{aligned}$$

Since $u, v \in [1, 3]$ and $q_1, q_2 \in (0, 1)$, all four mixed partial q -derivatives are positive on Δ . Therefore,

$$|\varphi_i(u, v)| = \varphi_i(u, v), \quad i = 1, 2, 3, 4.$$

Moreover,

$$\lim_{\substack{q_1 \rightarrow 1^- \\ q_2 \rightarrow 1^-}} \varphi_i(u, v) = 4uv = \frac{\partial^2 \Psi}{\partial u \partial v}(u, v), \quad i = 1, 2, 3, 4,$$

which agrees with the corresponding classical mixed partial derivative.

Example 4.2. Let $\Phi : \Delta = [1, 3] \times [1, 3] \rightarrow \mathbb{R}$ be defined by $\Phi(u, v) = u^3 v^3$. Using Definitions 2.13 and 2.14, we obtain

$$\begin{aligned}
\varphi_1(u, v) &= [u^2 + u(q_1 u + 1 - q_1) + (q_1 u + 1 - q_1)^2] \\
&\quad \times [v^2 + v(q_2 v + 1 - q_2) + (q_2 v + 1 - q_2)^2], \\
\varphi_2(u, v) &= [u^2 + u(q_1 u + 1 - q_1) + (q_1 u + 1 - q_1)^2] \\
&\quad \times [v^2 + v(q_2 v + 3(1 - q_2)) + (q_2 v + 3(1 - q_2))^2], \\
\varphi_3(u, v) &= [u^2 + u(q_1 u + 3(1 - q_1)) + (q_1 u + 3(1 - q_1))^2] \\
&\quad \times [v^2 + v(q_2 v + 1 - q_2) + (q_2 v + 1 - q_2)^2], \\
\varphi_4(u, v) &= [u^2 + u(q_1 u + 3(1 - q_1)) + (q_1 u + 3(1 - q_1))^2] \\
&\quad \times [v^2 + v(q_2 v + 3(1 - q_2)) + (q_2 v + 3(1 - q_2))^2].
\end{aligned}$$

Since $u, v \in [1, 3]$ and $q_1, q_2 \in (0, 1)$, all the factors in these expressions are positive. Therefore,

$$|\varphi_i(u, v)| = \varphi_i(u, v), \quad i = 1, 2, 3, 4.$$

Moreover, the mixed partial q -derivatives are continuous on the corresponding compact subrectangles and are therefore q -integrable. We next verify the coordinated strong s -convexity required in Theorem 3.1. Direct calculation gives

$$\begin{aligned}
\frac{d^2}{dx^2} [x^2 + x(qx + 1 - q) + (qx + 1 - q)^2] &= 2[3]_q, \\
\frac{d^2}{dx^2} [x^2 + x(qx + 3(1 - q)) + (qx + 3(1 - q))^2] &= 2[3]_q.
\end{aligned}$$

For every $x \in [1, 3]$ and $q \in (0, 1)$, the shifted arguments

$$qx + 1 - q \quad \text{and} \quad qx + 3(1 - q)$$

belong to $[1, 3]$. Since $x^2 + xz + z^2 \geq 3$ whenever $x, z \geq 1$, it follows that

$$\begin{aligned}
x^2 + x(qx + 1 - q) + (qx + 1 - q)^2 &\geq 3, \\
x^2 + x(qx + 3(1 - q)) + (qx + 3(1 - q))^2 &\geq 3.
\end{aligned}$$

Consequently, for $i = 1, 2, 3, 4$,

$$\frac{\partial^2 |\varphi_i|}{\partial u^2}(u, v) \geq 6[3]_{q_1} \geq 6$$

and

$$\frac{\partial^2 |\varphi_i|}{\partial v^2}(u, v) \geq 6[3]_{q_2} \geq 6.$$

By the second-derivative characterization of strong convexity, a twice differentiable function f is strongly convex with modulus

c whenever $f'' \geq 2c$. Thus, each $|\varphi_i|$ is coordinated strongly convex with respect to $(3, 3)$. Moreover, for $s \in (0, 1]$,

$$\tau^s \geq \tau, \quad (1 - \tau)^s \geq 1 - \tau, \quad \tau \in [0, 1].$$

Since $|\varphi_i| \geq 0$, the strong convexity inequality therefore implies the corresponding strong s -convexity inequality with the same moduli. Accordingly, each $|\varphi_i|$ is coordinated strongly s -convex with respect to $(3, 3)$ for every $s \in (0, 1]$.

Accordingly, we take

$$e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = e_7 = e_8 = 3.$$

We now evaluate the left-hand side of Theorem 3.1. First,

$$\begin{aligned} & \frac{\Phi(1, 1) + \Phi(1, 3) + \Phi(3, 1) + \Phi(3, 3)}{4} \\ &= \frac{1 + 27 + 27 + 729}{4} = 196. \end{aligned}$$

The four double q -integrals are

$$\begin{aligned} & \int_1^2 \int_1^2 u^3 v^3 {}_2d_{q_2} v {}_2d_{q_1} u \\ &= \left(8 - \frac{12}{[2]_{q_1}} + \frac{6}{[3]_{q_1}} - \frac{1}{[4]_{q_1}}\right) \\ & \quad \times \left(8 - \frac{12}{[2]_{q_2}} + \frac{6}{[3]_{q_2}} - \frac{1}{[4]_{q_2}}\right), \\ & \int_1^2 \int_2^3 u^3 v^3 {}_2d_{q_2} v {}_2d_{q_1} u \\ &= \left(8 - \frac{12}{[2]_{q_1}} + \frac{6}{[3]_{q_1}} - \frac{1}{[4]_{q_1}}\right) \\ & \quad \times \left(8 + \frac{12}{[2]_{q_2}} + \frac{6}{[3]_{q_2}} + \frac{1}{[4]_{q_2}}\right), \\ & \int_2^3 \int_1^2 u^3 v^3 {}_2d_{q_2} v {}_2d_{q_1} u \\ &= \left(8 + \frac{12}{[2]_{q_1}} + \frac{6}{[3]_{q_1}} + \frac{1}{[4]_{q_1}}\right) \\ & \quad \times \left(8 - \frac{12}{[2]_{q_2}} + \frac{6}{[3]_{q_2}} - \frac{1}{[4]_{q_2}}\right), \\ & \int_2^3 \int_2^3 u^3 v^3 {}_2d_{q_2} v {}_2d_{q_1} u \\ &= \left(8 + \frac{12}{[2]_{q_1}} + \frac{6}{[3]_{q_1}} + \frac{1}{[4]_{q_1}}\right) \\ & \quad \times \left(8 + \frac{12}{[2]_{q_2}} + \frac{6}{[3]_{q_2}} + \frac{1}{[4]_{q_2}}\right). \end{aligned}$$

Hence, their sum is

$$\left(16 + \frac{12}{[3]_{q_1}}\right) \left(16 + \frac{12}{[3]_{q_2}}\right).$$

The four boundary q -integrals are

$$\int_1^2 (\Phi(u, 1) + \Phi(u, 3)) {}_2d_{q_1} u$$

$$= 28 \left(8 - \frac{12}{[2]_{q_1}} + \frac{6}{[3]_{q_1}} - \frac{1}{[4]_{q_1}}\right),$$

$$\begin{aligned} & \int_2^3 (\Phi(u, 1) + \Phi(u, 3)) {}_2d_{q_1} u \\ &= 28 \left(8 + \frac{12}{[2]_{q_1}} + \frac{6}{[3]_{q_1}} + \frac{1}{[4]_{q_1}}\right), \end{aligned}$$

$$\begin{aligned} & \int_1^2 (\Phi(1, v) + \Phi(3, v)) {}_2d_{q_2} v \\ &= 28 \left(8 - \frac{12}{[2]_{q_2}} + \frac{6}{[3]_{q_2}} - \frac{1}{[4]_{q_2}}\right), \end{aligned}$$

$$\begin{aligned} & \int_2^3 (\Phi(1, v) + \Phi(3, v)) {}_2d_{q_2} v \\ &= 28 \left(8 + \frac{12}{[2]_{q_2}} + \frac{6}{[3]_{q_2}} + \frac{1}{[4]_{q_2}}\right). \end{aligned}$$

Therefore, the left-hand side of (3.1) becomes

$$\begin{aligned} L(q_1, q_2) &= \left| 196 + \frac{1}{4} \left(16 + \frac{12}{[3]_{q_1}}\right) \left(16 + \frac{12}{[3]_{q_2}}\right) \right. \\ & \quad \left. - 7 \left(16 + \frac{12}{[3]_{q_1}}\right) - 7 \left(16 + \frac{12}{[3]_{q_2}}\right) \right| \\ &= 36 \left(1 - \frac{1}{[3]_{q_1}}\right) \left(1 - \frac{1}{[3]_{q_2}}\right). \end{aligned} \quad (4.1)$$

The values of the mixed q -derivatives at the vertices are as follows:

$$\begin{aligned} |\varphi_1(1, 1)| &= 9, \\ |\varphi_1(1, 3)| &= 3(13 + 10q_2 + 4q_2^2), \\ |\varphi_1(3, 1)| &= 3(13 + 10q_1 + 4q_1^2), \\ |\varphi_1(3, 3)| &= (13 + 10q_1 + 4q_1^2) \\ & \quad \times (13 + 10q_2 + 4q_2^2); \end{aligned}$$

$$\begin{aligned} |\varphi_2(1, 1)| &= 3(13 - 14q_2 + 4q_2^2), \\ |\varphi_2(1, 3)| &= 81, \\ |\varphi_2(3, 1)| &= (13 + 10q_1 + 4q_1^2) \\ & \quad \times (13 - 14q_2 + 4q_2^2), \\ |\varphi_2(3, 3)| &= 27(13 + 10q_1 + 4q_1^2); \end{aligned}$$

$$\begin{aligned} |\varphi_3(1, 1)| &= 3(13 - 14q_1 + 4q_1^2), \\ |\varphi_3(1, 3)| &= 27(13 - 14q_1 + 4q_1^2), \\ |\varphi_3(3, 1)| &= 81, \\ |\varphi_3(3, 3)| &= 27(13 + 10q_2 + 4q_2^2); \end{aligned}$$

and

$$\begin{aligned} |\varphi_4(1, 1)| &= (13 - 14q_1 + 4q_1^2) \\ & \quad \times (13 - 14q_2 + 4q_2^2), \end{aligned}$$

$$|\varphi_4(1, 3)| = 27(13 - 14q_1 + 4q_1^2),$$

$$|\varphi_4(3, 1)| = 27(13 - 14q_2 + 4q_2^2),$$

$$|\varphi_4(3, 3)| = 729.$$

It follows that

$$\begin{aligned} |\varphi_1(1, 1)| + |\varphi_2(1, 3)| + |\varphi_3(3, 1)| + |\varphi_4(3, 3)| \\ = 9 + 81 + 81 + 729 = 900, \end{aligned}$$

$$\begin{aligned} |\varphi_1(1, 3)| + |\varphi_2(1, 1)| + |\varphi_3(3, 3)| + |\varphi_4(3, 1)| \\ = 780 - 120q_2 + 240q_2^2, \end{aligned}$$

$$\begin{aligned} |\varphi_1(3, 1)| + |\varphi_2(3, 3)| + |\varphi_3(1, 1)| + |\varphi_4(1, 3)| \\ = 780 - 120q_1 + 240q_1^2, \end{aligned}$$

and

$$\begin{aligned} |\varphi_1(3, 3)| + |\varphi_2(3, 1)| + |\varphi_3(1, 3)| + |\varphi_4(1, 1)| \\ = (26 - 4q_1 + 8q_1^2)(26 - 4q_2 + 8q_2^2). \end{aligned}$$

Since

$$(b_1 - a_1)(b_2 - a_2) = (3 - 1)(3 - 1) = 4$$

and

$$e_1 + e_3 + e_5 + e_7 = e_2 + e_4 + e_6 + e_8 = 12,$$

the right-hand side of Theorem 3.1 is

$$\begin{aligned} R_1(q_1, q_2, s) = & \frac{q_1 q_2}{2^{2s+2}} \left[900 \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) \right. \\ & + (780 - 120q_2 + 240q_2^2) \left(\frac{1}{[2]_{q_1}} - \frac{s}{[3]_{q_1}} \right) \\ & \times \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) \\ & + (780 - 120q_1 + 240q_1^2) \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \\ & \times \left(\frac{1}{[2]_{q_2}} - \frac{s}{[3]_{q_2}} \right) \\ & + (26 - 4q_1 + 8q_1^2)(26 - 4q_2 + 8q_2^2) \\ & \times \left. \left(\frac{1}{[2]_{q_1}} + \frac{1}{[s+2]_{q_1}} \right) \left(\frac{1}{[2]_{q_2}} + \frac{1}{[s+2]_{q_2}} \right) \right] \\ & - \frac{3q_1^3 q_2}{[2]_{q_2} [4]_{q_1}} - \frac{3q_1 q_2^3}{[2]_{q_1} [4]_{q_2}}. \end{aligned} \quad (4.2)$$

Hence, Theorem 3.1 yields

$$L(q_1, q_2) \leq R_1(q_1, q_2, s), \quad q_1, q_2 \in (0, 1), \quad s \in (0, 1].$$

Figure 1 illustrates the quantities $L(q_1, q_2)$ and $R_1(q_1, q_2, s)$ for $s = 1$ and $s = 0.5$.

The figure is provided only as a graphical illustration. The validity of the inequality follows from Theorem 3.1 and the verified coordinated strong s -convexity of $|\varphi_i|$.

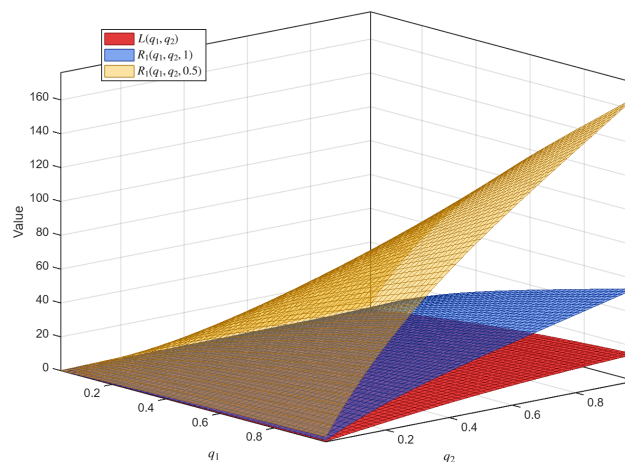


Figure 1: Graphical comparison of $L(q_1, q_2)$ with the upper bounds $R_1(q_1, q_2, 1)$ and $R_1(q_1, q_2, 0.5)$ on the tested parameter domain.

Example 4.3. Consider the function $\Phi(u, v) = u^3 v^3$, $(u, v) \in \Delta = [1, 3] \times [1, 3]$, as in Example 4.2. The left-hand side of Theorem 3.2 is

$$L(q_1, q_2) = 36 \left(1 - \frac{1}{[3]_{q_1}} \right) \left(1 - \frac{1}{[3]_{q_2}} \right). \quad (4.3)$$

We take

$$i_1 = i_2 = 2, \quad a_1 = a_2 = 1, \quad b_1 = b_2 = 3.$$

Hence,

$$\frac{(b_1 - a_1)(b_2 - a_2)}{16} = \frac{1}{4}$$

and

$$\left(\frac{q_1^{i_2} q_2^{i_2}}{[i_2 + 1]_{q_1} [i_2 + 1]_{q_2}} \right)^{1/i_2} = \frac{q_1 q_2}{\sqrt{[3]_{q_1} [3]_{q_2}}}.$$

We first verify the hypotheses of Theorem 3.2. Since the mixed partial q -derivatives are polynomials in u and v , their squares are q -integrable on Δ . From Example 4.2, for each $i = 1, 2, 3, 4$,

$$\varphi_i(u, v) \geq 9,$$

and

$$\frac{\partial^2 \varphi_i}{\partial u^2}(u, v) \geq 6, \quad \frac{\partial^2 \varphi_i}{\partial v^2}(u, v) \geq 6.$$

Since $\varphi_i > 0$, we have

$$|\varphi_i|^2 = \varphi_i^2.$$

Therefore,

$$\begin{aligned} \frac{\partial^2 |\varphi_i|^2}{\partial u^2} &= 2 \left(\frac{\partial \varphi_i}{\partial u} \right)^2 + 2 \varphi_i \frac{\partial^2 \varphi_i}{\partial u^2} \\ &\geq 2(9)(6) = 108, \end{aligned}$$

and, similarly,

$$\frac{\partial^2 |\varphi_i|^2}{\partial v^2} \geq 108.$$

Since $108 = 2(54)$, the second-derivative characterization shows that $|\varphi_i|^2$ is coordinated strongly convex with respect to $(54, 54)$. Moreover, $|\varphi_i|^2$ is nonnegative and $\tau^s \geq \tau$, $(1 - \tau)^s \geq 1 - \tau$ for $s \in (0, 1]$. Hence, $|\varphi_i|^2$ is coordinated strongly s -convex with respect to $(54, 54)$ for every $s \in (0, 1]$. Accordingly, we take

$$e_1 = e_2 = e_3 = e_4 = e_5 = e_6 = e_7 = e_8 = 54.$$

The squared vertex values required in Theorem 3.2 are

$$\begin{aligned} |\varphi_1(1, 1)|^2 &= 81, \\ |\varphi_1(1, 3)|^2 &= 9(13 + 10q_2 + 4q_2^2)^2, \\ |\varphi_1(3, 1)|^2 &= 9(13 + 10q_1 + 4q_1^2)^2, \\ |\varphi_1(3, 3)|^2 &= (13 + 10q_1 + 4q_1^2)^2 \\ &\quad \times (13 + 10q_2 + 4q_2^2)^2; \end{aligned}$$

$$\begin{aligned} |\varphi_2(1, 1)|^2 &= 9(13 - 14q_2 + 4q_2^2)^2, \\ |\varphi_2(1, 3)|^2 &= 6561, \\ |\varphi_2(3, 1)|^2 &= (13 + 10q_1 + 4q_1^2)^2 \\ &\quad \times (13 - 14q_2 + 4q_2^2)^2, \\ |\varphi_2(3, 3)|^2 &= 729(13 + 10q_1 + 4q_1^2)^2; \end{aligned}$$

$$\begin{aligned} |\varphi_3(1, 1)|^2 &= 9(13 - 14q_1 + 4q_1^2)^2, \\ |\varphi_3(1, 3)|^2 &= 729(13 - 14q_1 + 4q_1^2)^2, \\ |\varphi_3(3, 1)|^2 &= 6561, \\ |\varphi_3(3, 3)|^2 &= 729(13 + 10q_2 + 4q_2^2)^2; \end{aligned}$$

and

$$\begin{aligned} |\varphi_4(1, 1)|^2 &= (13 - 14q_1 + 4q_1^2)^2 \\ &\quad \times (13 - 14q_2 + 4q_2^2)^2, \\ |\varphi_4(1, 3)|^2 &= 729(13 - 14q_1 + 4q_1^2)^2, \\ |\varphi_4(3, 1)|^2 &= 729(13 - 14q_2 + 4q_2^2)^2, \\ |\varphi_4(3, 3)|^2 &= 531441. \end{aligned}$$

Substituting these values into Theorem 3.2, the first term inside the square brackets is

$$\begin{aligned} \mathcal{A}_1 &= \frac{1}{2^{2s}} \left[81 + 9 \left(1 + \frac{1}{[s+1]_{q_2}} \right) (13 + 10q_2 + 4q_2^2)^2 \right. \\ &\quad + 9 \left(1 + \frac{1}{[s+1]_{q_1}} \right) (13 + 10q_1 + 4q_1^2)^2 \\ &\quad + \left(1 + \frac{1}{[s+1]_{q_1}} \right) \left(1 + \frac{1}{[s+1]_{q_2}} \right) \\ &\quad \times (13 + 10q_1 + 4q_1^2)^2 (13 + 10q_2 + 4q_2^2)^2 \Big] \\ &\quad - 54 \frac{q_1[2]_{q_1}}{[3]_{q_1}} - 54 \frac{q_2[2]_{q_2}}{[3]_{q_2}}. \end{aligned}$$

The second term is

$$\mathcal{A}_2 = \frac{1}{2^{2s}} \left[9 \left(1 + \frac{1}{[s+1]_{q_2}} \right) (13 - 14q_2 + 4q_2^2)^2 \right.$$

$$\begin{aligned} &\quad + 6561 + \left(1 + \frac{1}{[s+1]_{q_1}} \right) \left(1 + \frac{1}{[s+1]_{q_2}} \right) \\ &\quad \times (13 + 10q_1 + 4q_1^2)^2 (13 - 14q_2 + 4q_2^2)^2 \\ &\quad + 729 \left(1 + \frac{1}{[s+1]_{q_1}} \right) (13 + 10q_1 + 4q_1^2)^2 \Big] \\ &\quad - 54 \frac{q_1[2]_{q_1}}{[3]_{q_1}} - 54 \frac{q_2[2]_{q_2}}{[3]_{q_2}}. \end{aligned}$$

The third term is

$$\begin{aligned} \mathcal{A}_3 &= \frac{1}{2^{2s}} \left[9 \left(1 + \frac{1}{[s+1]_{q_1}} \right) (13 - 14q_1 + 4q_1^2)^2 \right. \\ &\quad + 729 \left(1 + \frac{1}{[s+1]_{q_1}} \right) \left(1 + \frac{1}{[s+1]_{q_2}} \right) \\ &\quad \times (13 - 14q_1 + 4q_1^2)^2 \\ &\quad + 6561 + 729 \left(1 + \frac{1}{[s+1]_{q_2}} \right) (13 + 10q_2 + 4q_2^2)^2 \Big] \\ &\quad - 54 \frac{q_1[2]_{q_1}}{[3]_{q_1}} - 54 \frac{q_2[2]_{q_2}}{[3]_{q_2}}. \end{aligned}$$

Finally, the fourth term is

$$\begin{aligned} \mathcal{A}_4 &= \frac{1}{2^{2s}} \left[\left(1 + \frac{1}{[s+1]_{q_1}} \right) \left(1 + \frac{1}{[s+1]_{q_2}} \right) \right. \\ &\quad \times (13 - 14q_1 + 4q_1^2)^2 (13 - 14q_2 + 4q_2^2)^2 \\ &\quad + 729 \left(1 + \frac{1}{[s+1]_{q_1}} \right) (13 - 14q_1 + 4q_1^2)^2 \\ &\quad + 729 \left(1 + \frac{1}{[s+1]_{q_2}} \right) (13 - 14q_2 + 4q_2^2)^2 \\ &\quad + 531441 \Big] - 54 \frac{q_1[2]_{q_1}}{[3]_{q_1}} - 54 \frac{q_2[2]_{q_2}}{[3]_{q_2}}. \end{aligned}$$

Consequently, the right-hand side of Theorem 3.2 becomes

$$\begin{aligned} R_2(q_1, q_2, s) &= \frac{q_1 q_2}{4 \sqrt{[3]_{q_1}} [3]_{q_2}} \\ &\quad \times \left(\sqrt{\mathcal{A}_1} + \sqrt{\mathcal{A}_2} + \sqrt{\mathcal{A}_3} + \sqrt{\mathcal{A}_4} \right). \end{aligned} \quad (4.4)$$

Therefore, Theorem 3.2 gives

$$L(q_1, q_2) \leq R_2(q_1, q_2, s), \quad q_1, q_2 \in (0, 1), \quad s \in (0, 1].$$

Figure 2 illustrates the quantities $L(q_1, q_2)$, $R_2(q_1, q_2, 1)$, and $R_2(q_1, q_2, 0.5)$.

The figure is provided only as a graphical illustration. The validity of the inequality follows from Theorem 3.2 and the verified coordinated strong s -convexity of $|\varphi_i|^2$.

Example 4.4. Consider the function

$$\Phi(u, v) = u^3 v^3, \quad (u, v) \in \Delta = [1, 3] \times [1, 3],$$

as in Examples 4.2 and 4.3. Its four mixed partial q -derivatives exist on Δ° . Moreover, since these derivatives are polynomials

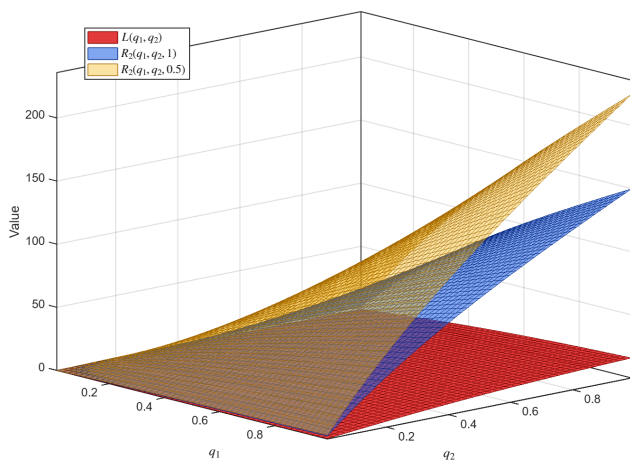


Figure 2: Graphical comparison of $L(q_1, q_2)$ with the upper bounds $R_2(q_1, q_2, 1)$ and $R_2(q_1, q_2, 0.5)$ on the tested parameter domain.

in u and v , their squares are q -integrable on Δ . The left-hand side of inequality (3.22) is

$$L(q_1, q_2) = 36 \left(1 - \frac{1}{[3]_{q_1}}\right) \left(1 - \frac{1}{[3]_{q_2}}\right). \quad (4.5)$$

To apply Theorem 3.3, set

$$i_1 = 2, \quad a_1 = a_2 = 1, \quad b_1 = b_2 = 3.$$

Thus,

$$1 - \frac{1}{i_1} = \frac{1}{2}, \quad \frac{(b_1 - a_1)(b_2 - a_2)}{16} = \frac{1}{4}.$$

As shown in Example 4.3, each

$$|\varphi_i|^2, \quad i = 1, 2, 3, 4,$$

is coordinated strongly s -convex with respect to (54, 54) for every $s \in (0, 1]$. Hence,

$$e_1 = e_2 = \dots = e_8 = 54.$$

For convenience, set

$$\alpha_j = \frac{1}{[2]_{q_j}} - \frac{s}{[3]_{q_j}}, \quad \beta_j = \frac{1}{[2]_{q_j}} + \frac{1}{[s+2]_{q_j}}, \quad j = 1, 2,$$

and

$$X_j = 13 + 10q_j + 4q_j^2, \quad Y_j = 13 - 14q_j + 4q_j^2, \quad j = 1, 2.$$

The squared vertex values required in Theorem 3.3 are

$$\begin{aligned} |\varphi_1(1, 1)|^2 &= 81, & |\varphi_1(1, 3)|^2 &= 9Y_1^2, \\ |\varphi_1(3, 1)|^2 &= 9X_1^2, & |\varphi_1(3, 3)|^2 &= X_1^2X_2^2; \end{aligned}$$

$$\begin{aligned} |\varphi_2(1, 1)|^2 &= 9Y_2^2, & |\varphi_2(1, 3)|^2 &= 6561, \\ |\varphi_2(3, 1)|^2 &= X_1^2Y_2^2, & |\varphi_2(3, 3)|^2 &= 729X_1^2; \end{aligned}$$

$$\begin{aligned} |\varphi_3(1, 1)|^2 &= 9Y_1^2, & |\varphi_3(1, 3)|^2 &= 729Y_1^2, \\ |\varphi_3(3, 1)|^2 &= 6561, & |\varphi_3(3, 3)|^2 &= 729X_2^2; \end{aligned}$$

and

$$\begin{aligned} |\varphi_4(1, 1)|^2 &= Y_1^2Y_2^2, & |\varphi_4(1, 3)|^2 &= 729Y_1^2, \\ |\varphi_4(3, 1)|^2 &= 729Y_2^2, & |\varphi_4(3, 3)|^2 &= 531441. \end{aligned}$$

Since

$$(a_1 - b_1)^2 = (a_2 - b_2)^2 = 4,$$

the first quantity on the right-hand side of Theorem 3.3 is

$$\begin{aligned} C_1 &= \frac{q_1q_2}{2^{2s}} \left(81\alpha_1\alpha_2 + 9X_2^2\alpha_1\beta_2 \right. \\ &\quad \left. + 9X_1^2\beta_1\alpha_2 + X_1^2X_2^2\beta_1\beta_2 \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right). \end{aligned}$$

The second quantity is

$$\begin{aligned} C_2 &= \frac{q_1q_2}{2^{2s}} \left(9Y_2^2\alpha_1\beta_2 + 6561\alpha_1\alpha_2 \right. \\ &\quad \left. + X_1^2Y_2^2\beta_1\beta_2 + 729X_1^2\beta_1\alpha_2 \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right). \end{aligned}$$

The third quantity is

$$\begin{aligned} C_3 &= \frac{q_1q_2}{2^{2s}} \left(9Y_1^2\beta_1\alpha_2 + 729Y_1^2\beta_1\beta_2 \right. \\ &\quad \left. + 6561\alpha_1\alpha_2 + 729X_2^2\alpha_1\beta_2 \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right). \end{aligned}$$

Finally, the fourth quantity is

$$\begin{aligned} C_4 &= \frac{q_1q_2}{2^{2s}} \left(Y_1^2Y_2^2\beta_1\beta_2 + 729Y_1^2\beta_1\alpha_2 \right. \\ &\quad \left. + 729Y_2^2\alpha_1\beta_2 + 531441\alpha_1\alpha_2 \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_2}} \left(\frac{1}{[2]_{q_1}} - \frac{1}{[4]_{q_1}} \right) \\ &\quad - \frac{54q_1q_2}{[2]_{q_1}} \left(\frac{1}{[2]_{q_2}} - \frac{1}{[4]_{q_2}} \right). \end{aligned}$$

Consequently, the right-hand side of Theorem 3.3 becomes

$$\begin{aligned} R_3(q_1, q_2, s) &= \frac{1}{4} \left(\frac{q_1q_2}{[2]_{q_1}[2]_{q_2}} \right)^{1/2} \\ &\quad \times (\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \sqrt{C_4}). \end{aligned} \quad (4.6)$$

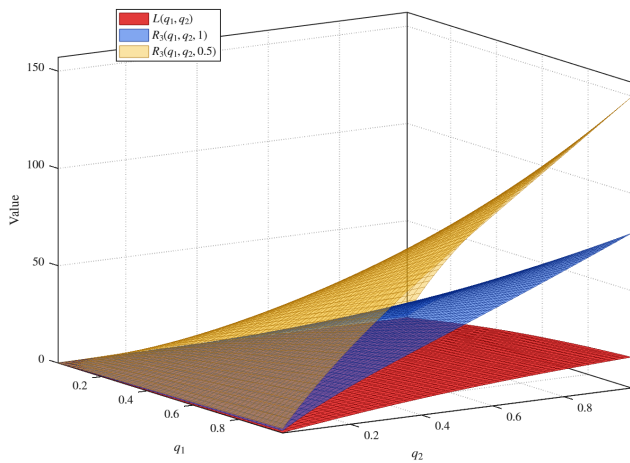


Figure 3: Graphical comparison of $L(q_1, q_2)$ with the upper bounds $R_3(q_1, q_2, 1)$ and $R_3(q_1, q_2, 0.5)$ on the tested parameter domain.

Table 1: Minimum numerical gaps on the 401×401 grid over $[0.01, 0.99]^2$.

Bound	s	Minimum gap	Location (q_1, q_2)
R_1	1	0.012917993379	(0.01, 0.01)
R_1	0.5	0.051277462362	(0.01, 0.01)
R_2	1	0.031986556559	(0.01, 0.01)
R_2	0.5	0.046740669420	(0.01, 0.01)
R_3	1	0.017299138977	(0.01, 0.01)
R_3	0.5	0.036536611710	(0.01, 0.01)

Hence,

$$L(q_1, q_2) \leq R_3(q_1, q_2, s), \quad q_1, q_2 \in (0, 1), \quad s \in (0, 1].$$

Figure 3 illustrates $L(q_1, q_2)$ together with the bounds $R_3(q_1, q_2, 1)$ and $R_3(q_1, q_2, 0.5)$.

The figure is included only as an illustration. The inequality follows from Theorem 3.3 and the verified coordinated strong s -convexity of $|\varphi_i|^2$.

4.1. Numerical procedure and minimum-gap analysis

For the numerical computations, the parameter domain was truncated to

$$[0.01, 0.99] \times [0.01, 0.99].$$

All calculations were performed in MATLAB using double-precision arithmetic on a uniform 401×401 grid, with grid spacing 0.00245 in each direction and numerical tolerance 10^{-12} . The minimum gap was determined by evaluating

$$R_i(q_1, q_2, s) - L(q_1, q_2), \quad i = 1, 2, 3,$$

at every grid point; thus, an exhaustive grid search was performed over all 401^2 parameter pairs.

No negative gap below the prescribed tolerance was detected. These computations provide finite-grid numerical checks and are not used as proofs over the continuous parameter domain. The MATLAB scripts used for the numerical computations and figures are provided as supplementary material.

Remark 4.1. The numerical computations yielded nonnegative gaps at all tested grid points for $s = 1$ and $s = 0.5$. These results illustrate the estimates established in Theorems 3.1–3.3; their validity follows from the corresponding theoretical results.

5. Conclusion

We established several quantum trapezoid-type inequalities for coordinated strongly s -convex functions on rectangles. The results were derived from mixed quantum integral identities using direct estimates, the quantum Hölder inequality, and the quantum power mean inequality. The symmetric modulus convention was applied consistently to both coordinates, and the corresponding classical inequalities were obtained as $q_1, q_2 \rightarrow 1^-$.

The assumptions of the main theorems were verified through explicit examples. Numerical computations on specified finite grids, together with the reported minimum gaps, illustrate the obtained bounds.

The approach may also be applied to quantum midpoint-, Simpson-, Ostrowski-, and Hermite–Hadamard-type inequalities for other classes of generalized convex functions.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

Funding

The authors received no specific funding from any public, commercial, or not-for-profit organisation for this research.

Acknowledgment

The authors would like to thank the anonymous reviewers and the Editor for their valuable comments and constructive suggestions, which helped improve the quality of this manuscript. The authors also acknowledge the support of Kalasin University, Thailand.

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