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Existence and Ulam–Hyers stability of nonlinear fractional differential equations via enriched (g, α, β, k)-functional contractions

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Abstract

This study investigates the existence, uniqueness, and Ulam–Hyers stability of solutions to nonlinear fractional differential equations (NFDEs) involving the Caputo fractional derivative. We extend and combine established fixed-point concepts into a class of contractive mappings on Banach spaces called enriched (g, α, β, k)-functional contractions. A bounded linear operator g supplements the metric structure, while the functional contraction rates α and β allow non-constant contraction behavior; a parameter k controls the weight of the auxiliary mapping. We prove a fixed-point theorem for this class and apply it to initial value problems for NFDEs. Under Lipschitz and continuity assumptions on the nonlinearity, we establish existence and uniqueness of a solution in a space of continuous functions. We also derive explicit sufficient conditions for Ulam–Hyers stability, together with a computable upper bound. The applicability of the results is discussed for fractional relaxation–oscillation models and fractional population dynamics, and several examples are used to examine the hypotheses and illustrate the theorems. The framework links fixed-point theory with the qualitative analysis of fractional systems through a unified approach to stability analysis.

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1. Introduction

Over the last 30 years, fractional calculus, which concerns derivatives and integrals of arbitrary (non-integer) order, has attracted considerable interest. Fractional operators are non-local and can incorporate memory effects and hereditary characteristics in complex physical, biological, and engineering systems. This property has motivated the use of fractional differential equations (FDEs) as powerful modeling tools. Whether in the study of the viscoelastic properties of polymers, the anomalous diffusion in porous media, the dynamics of biological populations, or the modelling of financial markets, FDEs have been invaluable. The rigorous mathematical foundations for this field have been provided by the seminal works of Kilbas [1], Podlubny [2] and Diethelm [3] and remain a solid foundation on which research continues to grow in this field.

At the same time the theory of fixed points has become one of the keystones of the branch of nonlinear analysis that is known as the study of differential and integral equations, and a very nice and unified approach to solving many such equations. The Banach contraction principle [4] is a classical result from 1922 which states that each contraction mapping on a complete

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metric space has exactly one fixed point. The theorem has been widely extended in a variety of ways, by applying it to more sophisticated mathematical frameworks. The Kannan fixed-point theorem [5] and the Chatterjea fixed-point theorem [6] were two of the more prominent generalizations which used the distances between the points and their images to shrink the distance between the images of two points. Recently, an alternative notion of *enriched contractions* was proposed in [7–11] for mappings that only weakly contract distances, with an additional parameter. At the same time, "functional contractions" were invented, which enabled the contraction constant to be different for different pairs of points thus extending the class of operators.

Another area of great interest has been the qualitative analysis of fractional differential equations, especially their stability. The idea of stability in the sense of Ulam and Hyers was first raised by Ulam in 1940 when he asked about the stability of group homomorphisms [12]. Hyers later gave his first positive solution to the additive Cauchy equation [13]. In the years to come, Ulam–Hyers stability has been proved for different functional, differential and integral equations. In the case of ODEs, the systematic criteria was laid out in the foundations of Rus [14]. The above ideas of stability are extended to fractional systems, but there are unique difficulties with fractional derivatives because they are non-local. The study of the stability of fractional differential equations under different conditions has been examined by Abbas *et al.* [15] while there is still much room for research on the relationship between the novel fixed-point techniques and Ulam–Hyers stability.

Although a large number of studies have been conducted, there is still a lack of studies. Enrichment of contractions, functional contractions, and the synthesis of either have been investigated separately, but the synthesis of enriched contractions with functional control parameters and/or contractions with an auxiliary operator g has not been extensively investigated. Moreover, the formulation of the existence and explicit Ulam–Hyers stability bounds of nonlinear fractional differential equations [16–22], using only such uniform contractive frameworks [23–25], is missing in the current literature. The majority of the existing literature are based on standard Banach contraction or Krasnoselskii-type fixed point arguments, leading to conservative stability bound or nonexplicit stability guarantees [26–31].

Inspired by these observations, the present paper aims to overcome these deficiencies by proposing a class of mappings different from those introduced above: the class of enriched (g, α, β, k)-functional contractions. The class is the product of three things: an auxiliary mapping g defined on the Banach space, the enrichment parameter k, and functional control functions α and β controlling the contraction behavior. We have a lot to offer: First, we present a complete fixed-point theorem for such mappings, establishing that under mild conditions, there exists a fixed point of the mapping, which is unique. Second, this abstract theorem is translated into concrete existence and uniqueness result for a class of Caputo-type nonlinear fractional differential equations. Third, and perhaps most significantly, we obtain explicit sufficient conditions for the Ulam–Hyers stability of these NFDEs, thus giving an explicit upper bound, depending on the problem parameters. The utility of our results is presented with real-world fractional models and the results are substantiated with a well-designed example.

This paper is organized as follows. In Section 2, we compile the required prerequisites: Definition from fractional calculus, fundamental concepts of fixed point and formal definition of our new contraction class. In Section 3, we state and prove the fixed-point theorem and its use in NFDEs. In Section 4, we delve into the Ulam-Hyers stability analysis, deriving the stability bound. In Section 5, we show how our results can be used in the context of fractional relaxation-oscillation equations and fractional logistic growth models. The sharpness and applicability of the theoretical conditions are illustrated by a number of numerical and analytical examples in Section 6. Finally, in Section 7, the concluding remarks and future research directions are given.

2. Preliminaries and mathematical framework

In order to continue our analysis, we must now set up the basic definitions, notations, and auxiliary lemmas that will be used in the manuscript. We shall assume, unless otherwise stated, that all spaces under consideration are real Banach spaces, endowed with a norm denoted by $\|\cdot\|$.

2.1. Fractional calculus essentials

Let $J = [0, T]$ for a fixed positive real number T (which may be ∞ in some theoretical settings, but we restrict ourselves to the finite interval for practical purposes). Let $\nu > 0$ be the order of the fractional derivative. We recall the definitions of the Riemann–Liouville fractional integral and the Caputo fractional derivative, which are the most commonly used operators in modeling physical phenomena.

Definition 2.1 (Riemann–Liouville Fractional Integral)**.** The Riemann–Liouville fractional integral of order $\nu > 0$ of a function $u : J \to \mathbb{R}$ is defined by

$$I^{\nu}u(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} u(s)\, ds,$$

provided that the right-hand side is pointwise defined on J. Here, $\Gamma(\cdot)$ denotes the Euler gamma function.

Definition 2.2 (Caputo Fractional Derivative)**.** The Caputo fractional derivative of order $\nu \in (n-1, n)$, $n \in \mathbb{N}$, of a function $u : J \to \mathbb{R}$ is defined by

$${}^{C}D^{\nu}u(t) = I^{n-\nu}u^{(n)}(t) = \frac{1}{\Gamma(n-\nu)} \int_0^t (t-s)^{n-\nu-1} u^{(n)}(s)\, ds,$$

where $u^{(n)}$ denotes the n-th ordinary derivative of u, provided the integral exists. In this paper, we are primarily concerned with the case where $\nu \in (0, 1)$, for which the definition simplifies to

$${}^{C}D^{\nu}u(t) = \frac{1}{\Gamma(1-\nu)} \int_0^t (t-s)^{-\nu} u'(s)\, ds.$$

The Caputo operator has a key property that makes it highly amenable to initial value problems: the derivative of a constant is zero, and the initial conditions involve integer-order derivatives, which are physically meaningful. The relationship between the integral and the derivative is captured by the following standard identity:

$$I^{\nu}\left({}^{C}D^{\nu}u(t)\right) = u(t) - u(0).$$

Let us define the space of continuous functions. Let $X = C(J,\mathbb{R})$ be the Banach space of all real-valued continuous functions defined on J, equipped with the supremum norm

$$\|u\|_{\infty} = \sup_{t\in J}|u(t)|.$$

It is well known that $(X,\|\cdot\|_{\infty})$ is complete, which is essential for employing fixed-point arguments. We will also make use of the space of continuously differentiable functions when required.

2.2. Fixed-point concepts and classical contractions

Before introducing our novel class of contractions, we briefly recall the standard definitions that motivate our work.

Definition 2.3 (Banach Contraction)**.** Let (X,d) be a complete metric space. A mapping $T : X \to X$ is called a Banach contraction if there exists a constant $0 \le \lambda < 1$ such that for all $x, y \in X$,

$$d(Tx,Ty) \le \lambda\, d(x,y).$$

Banach's theorem states that every Banach contraction on a complete metric space has a unique fixed point. This is the bedrock upon which most existence results in differential equations are built.

Definition 2.4 (Kannan Contraction)**.** A mapping $T : X \to X$ is a Kannan contraction if there exists $0 \le \beta < \frac{1}{2}$ such that for all $x, y \in X$,

$$d(Tx,Ty) \le \beta\left[d(x,Tx) + d(y,Ty)\right].$$

Definition 2.5 (Chatterjea Contraction)**.** A mapping $T : X \to X$ is a Chatterjea contraction if there exists $0 \le \gamma < \frac{1}{2}$ such that for all $x, y \in X$,

$$d(Tx,Ty) \le \gamma\left[d(x,Ty) + d(y,Tx)\right].$$

These generalizations of Banach's principle allow for contractions that do not rely solely on the distance between the preimages but also on the displacements of the points themselves. They have been extensively used to solve integral equations and differential equations.

Definition 2.6 (Enriched Contraction)**.** Let $(X,\|\cdot\|)$ be a Banach space. A mapping $T : X \to X$ is said to be an enriched contraction if there exist constants $b \ge 0$ and $\theta \in [0,1)$ such that for all $x, y \in X$,

$$\|Tx - Ty\| \le \theta\|x - y\| + b\|x - Tx\|.$$

This concept, introduced in [7], captures mappings where the contraction is aided by the discrepancy between the point and its image.

2.3. The new class: enriched (g,α,β,k)-functional contractions

We now introduce a new, highly general class of contractions that will serve as the core mathematical tool for our analysis. This class synthesizes the concepts of functional contractions (where the contraction factor is a function of the distance), enriched contractions (which incorporate the displacement term), and g-contractions (which involve an auxiliary operator).

Definition 2.7 (Enriched (g,α,β,k)-functional contraction)**.** Let (X,d) be a metric space and $T, g : X \to X$ be mappings. Let ψ be an altering distance function. We say that T is an enriched (g,α,β,k)-functional contraction if there exist constants $\alpha,\beta \ge 0$ with $\alpha+\beta < 1$, and a bounded function $k : X\times X \to [0,\infty)$ such that for all $x, y \in X$,

$$\psi(d(Tx,Ty)) \le \alpha\,\psi(d(gx,gy)) + \beta\,\psi(d(gx,Tx)) + k(x,y)\,\psi(d(gy,Ty)).$$

Moreover, we require that $k(x,y)$ satisfies $\sup_{x,y\in X} k(x,y) = K < \infty$, and we define the total contraction coefficient $\Lambda = \alpha+\beta+K < 1$.

Several remarks regarding Definition 2.7 are in order.

Remark 2.8 (Special cases and relation with classical contractions)**.** The relationship between Definition 2.7 and classical contraction conditions requires some care because the additional term $k\|gx - gy\|$ appears on the left-hand side of (2.1).

First, the classical Banach contraction principle is recovered by taking $g = 0$, $\beta(t) \equiv 0$, and

$$\alpha(t) \equiv \lambda, \qquad 0 \le \lambda < 1.$$

Indeed, in this case, condition (2.1) becomes

$$\|Tx - Ty\| \le \lambda\|x - y\|, \qquad x, y \in X,$$

which is precisely the classical Banach contraction condition.

In contrast, the choice $g = I$ and $k = 1$ does not recover the Banach contraction condition. In fact, with $\alpha(t) \equiv \lambda < 1$ and $\beta(t) \equiv 0$, condition (2.1) becomes

$$\|Tx - Ty\| + \|x - y\| \le \lambda\|x - y\|,$$

which cannot hold for $x \ne y$, since

$$\|Tx - Ty\| + \|x - y\| \ge \|x - y\| > \lambda\|x - y\|.$$

Therefore, $g = I$ is not an admissible choice for recovering the classical Banach contraction under these parameter values.

Similarly, the standard Kannan contraction is not obtained directly from Definition 2.7. If $g = 0$, $\alpha \equiv 0$, and $\beta(t) \equiv \beta_0$, condition (2.1) reduces to the one-sided condition

$$\|Tx - Ty\| \le \beta_0\|x - Tx\|,$$

which should be regarded as a Kannan-type condition rather than the standard Kannan contraction

$$\|Tx - Ty\| \le \beta_0(\|x - Tx\| + \|y - Ty\|).$$

Thus, the present framework contains the Banach contraction principle as a special case and includes a one-sided Kannan-type condition, but the standard Kannan contraction is not claimed as a direct special case of Definition 2.7.

Thus, the proposed definition provides a unified framework that incorporates the classical Banach contraction principle as a special case and introduces additional operator-dependent and distance-dependent contractive structures.

Remark 2.9 (Role of the auxiliary operator g)**.** The auxiliary operator g serves a dual purpose in our framework. First (Relaxation): If g is a contraction with Lipschitz constant $\lambda < 1$, then $d(gx, gy) \leq \lambda d(x, y)$. Substituting this into Definition 2.7 yields an effective contraction coefficient $\Lambda_{\text{eff}} = \alpha\lambda + \beta + K$. Since $\lambda < 1$, the condition $\Lambda_{\text{eff}} < 1$ permits α to be strictly greater than $1 - \beta - K$. Consequently, g *relaxes* the contractive requirement, allowing T to be non-contractive in the standard Banach sense (i.e., $\alpha \geq 1$ is admissible if λ is sufficiently small). Second (Error Cancellation): The terms $d(gx, Tx)$ and $d(gy, Ty)$ quantify the deviation between g and T. If g is chosen such that $g \approx T$, these terms vanish, effectively reducing the contribution of β and K to the overall contraction.

Remark 2.10. The role of the auxiliary operator g should be understood as an additional structural component of the contractive condition rather than as a direct relaxation of the classical Banach contraction inequality. In particular, the term $k\|gx - gy\|$ is non-negative and therefore makes the inequality more restrictive for a fixed choice of α, β, and k. The novelty of Definition 2.7 does not, therefore, consist in simply weakening the Banach contraction condition. Instead, it lies in combining, within a single contractive framework, an auxiliary operator g, the enrichment parameter k, and distance-dependent control functions α and β. This formulation provides a unified setting in which the contraction behavior may depend on the geometry induced by g and may vary with the distance between points. Consequently, the proposed framework should be viewed as a structured generalization and unification of several contractive conditions rather than as a uniformly weaker condition than the classical Banach contraction.

Remark 2.11. The condition $\|g\| \leq 1$ is crucial. It guarantees that $\|gx - gy\| \leq \|x - y\|$, and this helps in effectively bounding the left-hand side of (2.1). The fixed-point results might fail without further assumptions if $\|g\| > 1$.

Remark 2.12 (Admissible parameter region)**.** To ensure the existence of a unique fixed point, the total contraction coefficient must satisfy:

$$\Lambda_{\text{eff}} = \alpha\lambda + \beta + K < 1,$$

where λ is the Lipschitz constant of g and $K = \sup k(x, y)$.

The admissible parameter region is as follows:

- If $\lambda = 1$ (i.e., g is an isometry), the condition reduces to $\alpha + \beta + K < 1$. This is the classical strict contraction requirement.
- If $\lambda < 1$, the condition becomes $\alpha < \frac{1-\beta-K}{\lambda}$. Since the denominator is less than 1, the upper bound on α is strictly greater than $1 - \beta - K$. Thus, g *relaxes* the restriction on the main contraction coefficient.

Therefore, the statement in Remark 2.9 is consistent: g ensures convergence for a wider class of operators T than those satisfying the classical strict contraction condition.

In the first step, we would like to develop a fixed-point theorem for mappings fulfilling Definition 2.7. After doing this, we test it for the integral operator that was obtained from the fractional differential equation. Both existence and stability results will be obtained by this approach.

2.4. Ulam–Hyers stability

There are several aspects to the concept of stability with respect to differential equations. The Ulam-Hyers stability has become more popular because of its simplicity and practicality and we shall concentrate in this work on that. It is a way of quantifying the accuracy of an approximate solution with respect to an exact solution.

Definition 2.13 (Ulam–Hyers Stability)**.** Consider the fractional initial value problem

$$^{C}D^{\nu}u(t) = f(t, u(t)), \qquad t \in J = [0, T], \qquad u(0) = u_0, \quad (1)$$

where $0 < \nu < 1$. The problem (1) is said to be *Ulam–Hyers stable* if there exists a constant $C > 0$ such that, for every $\varepsilon > 0$ and every function $u \in C(J, \mathbb{R})$ satisfying

$$\left|^{C}D^{\nu}u(t) - f(t, u(t))\right| \leq \varepsilon, \qquad t \in J, \quad (2)$$

together with the same initial condition

$$u(0) = u_0, \quad (3)$$

there exists an exact solution $u^* \in C(J, \mathbb{R})$ of (1) such that

$$\|u - u^*\|_\infty \leq C\varepsilon. \quad (4)$$

The constant C is called a Ulam–Hyers stability constant. In the present work, we derive an explicit upper bound for such a constant.

The value of the constant C is known as the Ulam-Hyers stability constant. We will derive in our analysis an explicit expression for C as a function of the parameters of our enriched contraction, so as to get a quantitative guarantee of stability.

2.5. Auxiliary lemmas

Lemma 2.14. A few facts, well known, are stated which are to be repeated as the main theorems are approached.

Let $T(t) > 0$ be continuous on an interval $[a, b]$.Suppose $T(t) > 0$ is continuous, over an interval $[a, b]$.

Let $y, h \in C(J, \mathbb{R}^+)$ and $K \geq 0$ be a constant. If

$$y(t) \leq h(t) + K\int_0^t y(s)\,ds, \quad \text{for all } t \in J,$$

then

$$y(t) \le h(t) + K \int_0^t e^{K(t-s)} h(s)\, ds.$$

In particular, if $h(t)$ is a constant H, then $y(t) \le He^{Kt}$.

The standard Gronwall inequality is so widely used that we will also make use of a fractional variant, which is less often used but is also very useful for our purposes.

Lemma 2.15 (Fractional Gronwall Inequality)**.** Let $y \in C(J, \mathbb{R}^+)$ and suppose there exist constants $M \ge 0$ and $\nu > 0$ such that

$$y(t) \le M \int_0^t (t-s)^{\nu-1} y(s)\, ds, \quad t \in J.$$

Then $y(t) \equiv 0$ on J.

The classical Gronwall inequality for the derivative of the fractional integral is directly applicable to this lemma. It will be very helpful in demonstrating uniqueness of solutions.

3. Main results: fixed-point theorem and existence for NFDEs

In this section, we present the core theoretical contributions of our paper. We begin by establishing a rigorous fixed-point theorem for enriched (g, α, β, k)-functional contractions. Then, we carefully translate this abstract result into an existence and uniqueness theorem for a nonlinear fractional differential equation.

3.1. Fixed-point theorem for enriched (g, α, β, k)-functional contractions

Theorem 3.1. *Let $(X, \|\cdot\|)$ be a Banach space and let $T : X \to X$ be an enriched (g, α, β, k)-functional contraction as in Definition 2.7. Assume, in addition, that there exists a constant $q \in [0, 1)$ such that*

$$\frac{\alpha(t)}{1 - \beta(t)} \le q, \qquad \text{for all } t \ge 0. \tag{3.1}$$

Then T has a unique fixed point $x^* \in X$. Moreover, for every $x_0 \in X$, the Picard sequence

$$x_{n+1} = Tx_n, \qquad n = 0, 1, 2, \dots,$$

converges to x^*.

Proof. Let $x_0 \in X$ be arbitrary and define the Picard sequence

$$x_{n+1} = Tx_n, \qquad n \ge 0.$$

If $x_n = x_{n+1}$ for some n, then x_n is a fixed point of T. Hence, we may assume that

$$x_{n+1} \ne x_n, \qquad n \ge 0.$$

Set

$$d_n = \|x_{n+1} - x_n\|, \qquad n \ge 0.$$

Applying the contractive condition in Definition 2.7 to $x = x_n$ and $y = x_{n-1}$, for $n \ge 1$, gives

$$\|Tx_n - Tx_{n-1}\| + k\|gx_n - gx_{n-1}\| \le \alpha(d_{n-1})d_{n-1} + \beta(d_{n-1})\|x_n - Tx_n\|.$$

Since

$$Tx_n = x_{n+1}, \qquad Tx_{n-1} = x_n, \qquad \|x_n - Tx_n\| = d_n,$$

we obtain

$$d_n + k\|gx_n - gx_{n-1}\| \le \alpha(d_{n-1})d_{n-1} + \beta(d_{n-1})d_n.$$

Since the second term on the left-hand side is non-negative,

$$d_n \le \alpha(d_{n-1})d_{n-1} + \beta(d_{n-1})d_n.$$

Consequently,

$$(1 - \beta(d_{n-1}))d_n \le \alpha(d_{n-1})d_{n-1}.$$

Because $\beta(t) < 1$ for every $t \ge 0$, we have

$$d_n \le \frac{\alpha(d_{n-1})}{1 - \beta(d_{n-1})} d_{n-1}.$$

By the additional assumption (3.1),

$$d_n \le q d_{n-1}.$$

Therefore, by induction,

$$d_n \le q^n d_0, \qquad n \ge 0. \tag{3.2}$$

Since $0 \le q < 1$, it follows that

$$\sum_{n=0}^{\infty} d_n \le d_0 \sum_{n=0}^{\infty} q^n = \frac{d_0}{1-q} < \infty. \tag{3.3}$$

We now prove that (x_n) is a Cauchy sequence. For $m > n$, the triangle inequality and (3.2) give

$$\|x_m - x_n\| \le \sum_{i=n}^{m-1} \|x_{i+1} - x_i\| = \sum_{i=n}^{m-1} d_i \le d_0 \sum_{i=n}^{m-1} q^i.$$

Hence,

$$\|x_m - x_n\| \le \frac{d_0 q^n}{1-q}. \tag{3.4}$$

Since $q \in [0, 1)$, the right-hand side of (3.4) tends to zero as $n \to \infty$. Thus, (x_n) is a Cauchy sequence. Since X is complete, there exists $x^* \in X$ such that

$$x_n \longrightarrow x^*. \tag{3.5}$$

It remains to prove that x^* is a fixed point of T. Apply the contractive condition to $x = x_n$ and $y = x^*$. Since

$$\|x_n - Tx_n\| = d_n,$$

we obtain

$$\|Tx_n - Tx^*\| + k\|gx_n - gx^*\| \le \alpha(\|x_n - x^*\|)\|x_n - x^*\| + \beta(\|x_n - x^*\|)d_n.$$

Dropping the non-negative second term on the left-hand side yields

$$\|Tx_n - Tx^*\| \leq \alpha(\|x_n - x^*\|)\|x_n - x^*\| + \beta(\|x_n - x^*\|)d_n.$$

Because $0 \leq \alpha(t) < 1$ and $0 \leq \beta(t) < 1$, we have

$$\|Tx_n - Tx^*\| \leq \|x_n - x^*\| + d_n.$$

By (3.2) and (3.5),

$$\|Tx_n - Tx^*\| \longrightarrow 0.$$

Since $Tx_n = x_{n+1} \to x^*$, it follows that

$$Tx^* = x^*.$$

Thus, x^* is a fixed point of T.

Finally, we prove uniqueness. Suppose that x^* and y^* are two fixed points of T with $x^* \neq y^*$. Applying Definition 2.7 to $x = x^*$ and $y = y^*$ gives

$$\|x^* - y^*\| + k\|gx^* - gy^*\| \leq \alpha(\|x^* - y^*\|)\|x^* - y^*\|,$$

because

$$\|x^* - Tx^*\| = 0.$$

Therefore,

$$\|x^* - y^*\| \leq \alpha(\|x^* - y^*\|)\|x^* - y^*\|.$$

Since $x^* \neq y^*$, division by $\|x^* - y^*\|$ gives

$$1 \leq \alpha(\|x^* - y^*\|),$$

which contradicts $\alpha(t) < 1$ for every $t \geq 0$. Hence,

$$x^* = y^*.$$

Therefore, T has a unique fixed point, and the Picard sequence converges to this fixed point for every initial point $x_0 \in X$. □

Remark 3.2. The additional uniform condition

$$\sup_{t\geq 0} \frac{\alpha(t)}{1-\beta(t)} \leq q < 1$$

is imposed to guarantee a uniform geometric decay of the successive Picard differences. Indeed, it yields

$$\|x_{n+1} - x_n\| \leq q^n \|x_1 - x_0\|,$$

which ensures the summability of the successive differences and, consequently, the Cauchy property of the Picard sequence. No continuity assumption on the control functions α and β is required.

Remark 3.3. The above Theorem shows that the additional structure of g and k does not destroy the fixed point property; in fact, it allows for a flexible structure that may be shared with operators which are not the classical sense strict contractions. The proof relies on the condition that $\alpha(t) + \beta(t) < 1$ so that the contraction factor for the iterates of the sequence stays strictly below 1.

3.2. Existence and uniqueness for nonlinear fractional differential equations

With the fixed-point theorem in hand, we now turn our attention to the primary objective: establishing the existence and uniqueness of solutions for the following nonlinear fractional differential equation with a Caputo derivative:

$$\begin{cases} {}^C D^\nu u(t) = f(t, u(t)), & t \in J = [0, T], \quad \nu \in (0, 1), \\ u(0) = u_0, \end{cases} \tag{3.2}$$

where $f : J \times \mathbb{R} \to \mathbb{R}$ is a given continuous function. We assume that f satisfies a Lipschitz condition with respect to the second variable, which is a standard requirement for obtaining uniqueness.

(H1) There exists a constant $L > 0$ such that for all $t \in J$ and all $x, y \in \mathbb{R}$,

$$|f(t, x) - f(t, y)| \leq L|x - y|.$$

(H2) The function $t \mapsto f(t, 0)$ is bounded on J.

We note that assumption (H2) ensures that f is bounded over any bounded subset of $J \times \mathbb{R}$, which is helpful for the existence of solutions but not strictly necessary for the contraction argument. Under these assumptions, we convert the fractional differential equation into an equivalent integral equation.

Lemma 3.4 (Equivalence of the Integral Equation)**.** A function $u \in C(J, \mathbb{R})$ is a solution of (3.2) if and only if it satisfies the following Volterra integral equation:

$$u(t) = u_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t - s)^{\nu-1} f(s, u(s))\, ds, \quad t \in J. \tag{3.3}$$

Proof. The proof follows from the standard property of the Caputo derivative: applying the Riemann–Liouville integral of order ν to both sides of (3.2) yields $I^\nu({}^C D^\nu u)(t) = I^\nu f(t, u(t))$. Using the identity $I^\nu({}^C D^\nu u)(t) = u(t) - u(0)$, we immediately obtain (3.3). The converse is obtained by differentiating (3.3) using the Caputo derivative, which is a standard result in fractional calculus. □ □

Thus, the problem reduces to finding a fixed point of the operator $\mathcal{T} : C(J, \mathbb{R}) \to C(J, \mathbb{R})$ defined by

$$(\mathcal{T}u)(t) = u_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t - s)^{\nu-1} f(s, u(s))\, ds. \tag{3.4}$$

We will show that $\mathcal{T}$ satisfies the conditions of Theorem 3.1, provided that the constants in the problem satisfy a certain inequality.

Theorem 3.5. *Assume that the hypotheses* (H1) *and* (H2) *hold. Let*

$$M = \frac{LT^\nu}{\Gamma(\nu + 1)}.$$

Suppose that there exist a constant $0 < \lambda < 1$ *and an enrichment parameter* $k \geq 1$ *such that*

$$M + k\lambda < 1. \tag{3.5}$$

Then the fractional differential equation

$$\begin{cases} {}^C D^{\nu} u(t) = f(t, u(t)), \quad t \in J = [0, T], \quad \nu \in (0, 1), \\ u(0) = u_0, \end{cases}$$

has a unique solution $u^* \in C(J, \mathbb{R})$.

Proof. Let

$$X = C(J, \mathbb{R})$$

be equipped with the supremum norm

$$\|u\|_\infty = \sup_{t \in J} |u(t)|.$$

Then X is a Banach space. The fractional differential equation is equivalent to the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, u(s))\, ds.$$

Accordingly, define the operator $T : X \to X$ by

$$(Tu)(t) = u_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s, u(s))\, ds.$$

For $u, v \in X$, using the Lipschitz condition *(H1)*, we obtain

$$\begin{aligned} |(Tu)(t) - (Tv)(t)| &\leq \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |f(s, u(s)) - f(s, v(s))|\, ds \\ &\leq \frac{L}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} |u(s) - v(s)|\, ds \\ &\leq \frac{Lt^{\nu}}{\Gamma(\nu+1)} \|u - v\|_\infty. \end{aligned}$$

Taking the supremum over $t \in [0, T]$, we get

$$\|Tu - Tv\|_\infty \leq \frac{LT^{\nu}}{\Gamma(\nu+1)} \|u - v\|_\infty = M\|u - v\|_\infty. \tag{3.6}$$

Now define the auxiliary operator

$$gu = \lambda u, \qquad u \in X,$$

where $0 < \lambda < 1$. Then g is a bounded linear operator and

$$\|g\| = \lambda \leq 1.$$

Consequently,

$$\|gu - gv\|_\infty = \lambda \|u - v\|_\infty. \tag{3.7}$$

Combining (3.6) and (3.7), we obtain

$$\begin{aligned} \|Tu - Tv\|_\infty + k\|gu - gv\|_\infty &\leq M\|u - v\|_\infty + k\lambda\|u - v\|_\infty \\ &= (M + k\lambda)\|u - v\|_\infty. \end{aligned}$$

Set

$$\alpha(t) = \alpha_0 := M + k\lambda, \qquad \beta(t) = \beta_0 := 0, \qquad t \geq 0.$$

By assumption (3.5),

$$0 \leq \alpha_0 < 1, \qquad \alpha_0 + \beta_0 = M + k\lambda < 1.$$

Moreover, α is non-decreasing. Therefore,

$$\begin{aligned} \|Tu - Tv\|_\infty + k\|gu - gv\|_\infty \leq \alpha(\|u - v\|_\infty)\|u - v\|_\infty \\ + \beta(\|u - v\|_\infty)\|u - Tu\|_\infty. \end{aligned}$$

Hence, T is an enriched (g, α, β, k)-functional contraction in the sense of Definition 2.7.

By Theorem 3.1, T has a unique fixed point $u^* \in X$. Since $u^* = Tu^*$, the fixed-point equation is precisely the Volterra integral equation equivalent to the original fractional differential equation. Therefore, u^* is the unique solution of the fractional differential equation. □

Remark 3.6. The condition

$$M + k\lambda < 1 \tag{3.5}$$

is a sufficient condition specific to the proposed enriched (g, α, β, k)-functional contraction framework. Since $k \geq 1$ and $\lambda \geq 0$, we have $k\lambda \geq 0$, and hence

$$M + k\lambda < 1 \quad \Longrightarrow \quad M < 1.$$

Therefore, condition (3.5) does not enlarge the admissible parameter region of the classical Banach contraction condition $M < 1$. In fact, when $\lambda > 0$, the condition $M + k\lambda < 1$ is more restrictive than $M < 1$. The role of the auxiliary operator g in the present formulation should therefore not be interpreted as allowing the case $M \geq 1$. Rather, it provides an additional structural component of the contractive condition and allows the action of g to be incorporated together with the functional control parameters α and β.

Corollary 3.7. If g is taken as the zero operator, that is, $\lambda = 0$, and $k = 1$, then Theorem 3.4 reduces to the classical existence result for the fractional differential equation under the condition $M < 1$. Thus, the classical Banach contraction case is recovered as a special case of the proposed framework.

4. Ulam–Hyers stability analysis

Having established the existence and uniqueness of solutions, we now proceed to investigate the stability of the fractional differential equation (3.2) in the sense of Ulam and Hyers. This is a critical aspect of qualitative analysis, as it ensures that small perturbations in the equation or the initial data lead to small deviations in the solution.

Theorem 4.1. *Under the hypotheses of Theorem 3.2, the fractional differential equation (3.2) is Ulam–Hyers stable. Specifically, there exists a constant* $C > 0$ *such that for any* $\varepsilon > 0$ *and any function* $u \in C(J, \mathbb{R})$ *satisfying the inequality*

$$\|{}^C D^{\nu} u(t) - f(t, u(t))\| \leq \varepsilon, \quad t \in J, \tag{4.1}$$

there exists a unique solution $u^* \in C(J, \mathbb{R})$ *of (3.2) such that*

$$\|u - u^*\|_\infty \leq C\varepsilon. \tag{4.2}$$

Moreover, the stability constant C can be explicitly given by

$$C = \frac{(1+\beta_0)\delta}{1-\alpha_0},$$

where α_0 and β_0 are the constant control functions chosen in the proof of Theorem 3.2, and $\delta = \frac{T^\nu}{\Gamma(\nu+1)}$.

Proof. Let $\varepsilon > 0$ be given. Suppose $u \in C(J, \mathbb{R})$ is an ε-approximate solution, meaning it satisfies (4.1). We need to find the exact solution u^* of (3.2) and bound the distance between u and u^*.

First, we note that the exact solution u^* is the fixed point of the operator $\mathcal{T}$ defined in (3.4). Thus, $u^* = \mathcal{T}u^*$. Our goal is to estimate $\|u - u^*\|_\infty$. By the triangle inequality,

$$\|u - u^*\|_\infty \leq \|u - \mathcal{T}u\|_\infty + \|\mathcal{T}u - \mathcal{T}u^*\|_\infty. \tag{4.3}$$

We begin by estimating the first term, $\|u - \mathcal{T}u\|_\infty$. Since u is an approximate solution, we have:

$$\|{}^C D^\nu u(t) - f(t, u(t))\| \leq \varepsilon.$$

Applying the Riemann–Liouville integral I^ν to both sides, we obtain for $t \in J$:

$$\left| I^\nu({}^C D^\nu u)(t) - I^\nu f(t, u(t)) \right| \leq I^\nu(\varepsilon) = \frac{\varepsilon t^\nu}{\Gamma(\nu+1)} \leq \frac{\varepsilon T^\nu}{\Gamma(\nu+1)}.$$

Using the identity $I^\nu({}^C D^\nu u)(t) = u(t) - u_0$ and the definition of $\mathcal{T}$, we recognize that $\mathcal{T}u(t) = u_0 + I^\nu f(t, u(t))$. Therefore,

$$|u(t) - \mathcal{T}u(t)| \leq \frac{\varepsilon T^\nu}{\Gamma(\nu+1)}.$$

Taking the supremum over $t \in J$, we get:

$$\|u - \mathcal{T}u\|_\infty \leq \frac{T^\nu}{\Gamma(\nu+1)}\varepsilon = \delta\varepsilon. \tag{4.4}$$

Next, we estimate the second term in (4.3), $\|\mathcal{T}u - \mathcal{T}u^*\|_\infty$. Since $\mathcal{T}$ is an enriched (g, α, β, k)-functional contraction, we apply (3.6) with $x = u$ and $y = u^*$. Note that $\mathcal{T}u^* = u^*$. Thus:

$$\|\mathcal{T}u - \mathcal{T}u^*\|_\infty + k\|gu - gu^*\|_\infty \leq \alpha_0\|u - u^*\|_\infty + \beta_0\|u - \mathcal{T}u\|_\infty.$$

Since $k\|gu - gu^*\|_\infty \geq 0$, we can drop it to obtain:

$$\|\mathcal{T}u - \mathcal{T}u^*\|_\infty \leq \alpha_0\|u - u^*\|_\infty + \beta_0\|u - \mathcal{T}u\|_\infty. \tag{4.5}$$

Substituting (4.4) into (4.5), we have:

$$\|\mathcal{T}u - \mathcal{T}u^*\|_\infty \leq \alpha_0\|u - u^*\|_\infty + \beta_0\delta\varepsilon. \tag{4.6}$$

Now, plug (4.4) and (4.6) back into (4.3):

$$\|u - u^*\|_\infty \leq \delta\varepsilon + \alpha_0\|u - u^*\|_\infty + \beta_0\delta\varepsilon.$$

Rearranging the terms to isolate $\|u - u^*\|_\infty$, we get:

$$(1 - \alpha_0)\|u - u^*\|_\infty \leq (1 + \beta_0)\delta\varepsilon.$$

Since $\alpha_0 < 1$, we can divide by $1 - \alpha_0 > 0$:

$$\|u - u^*\|_\infty \leq \frac{(1+\beta_0)\delta}{1-\alpha_0}\varepsilon.$$

Let $C = \frac{(1+\beta_0)\delta}{1-\alpha_0}$. This constant is finite and positive. Thus, we have established the desired inequality:

$$\|u - u^*\|_\infty \leq C\varepsilon.$$

It is shown that the fractional differential equation is Ulam-Hyers stable. □ □

Remark 4.2. The quantity

$$C_{\mathrm{bd}} = \frac{(1+\beta_0)\delta}{1-\alpha_0}, \qquad \delta = \frac{T^\nu}{\Gamma(\nu+1)},$$

provides an explicit and computable upper bound for the Ulam–Hyers stability constant. It depends only on the problem parameters T, ν, and the selected control parameters α_0 and β_0. The present analysis establishes the validity of this bound but does not assert its optimality or sharpness. Establishing the smallest possible stability constant would require a separate optimality argument.

Remark 4.3. The proof of Theorem 4.1 uses the result that the approximate solution u is a solution to a perturbed version of the integral equation. Importantly, we did not need the perturbation function to be continuous or to have any regularity conditions other than those suggested in (4.1). This is an illustration of the strength of the Ulam–Hyers stability concept.

Corollary 4.4. If $\beta_0 = 0$, then the stability constant reduces to $C = \frac{\delta}{1-\alpha_0}$. In the case of a standard Banach contraction where $\alpha_0 = M$ (with $\beta_0 = 0$), the constant becomes $C = \frac{\delta}{1-M}$. This corresponds to the classical result for the stability of Volterra integral equations, and hence, we are seeing that our framework is able to appropriately extend the previous ones.

5. Applications to physical and biological models

The theoretical results obtained in the previous sections are not simply mathematical forms but have important implications for the study of real dynamical systems described by fractional differential equations. Here we show two important applications: fractional relaxation-oscillation and a fractional logistic population model. The examples illustrate the use of our enhanced contraction framework in guaranteeing existence and stability.

5.1. Fractional relaxation–oscillation model

The relaxation-oscillation equation is a classical model of viscoelasticity and electrical circuits. This model can be a fractional model which includes some memory in the material or circuit. The initial value problem is:

$$\begin{cases} {}^C D^\nu u(t) = -au(t) + b\sin(\omega t), & t \in [0, T], \quad \nu \in (0, 1), \\ u(0) = u_0, \end{cases} \tag{5.1}$$

where $a > 0$, $b \in \mathbb{R}$, and $\omega > 0$ are constants. The equation represents a system with fractional damping, and a sinusoidal forcing function.

The function $f(t,u) = -au + b\sin(\omega t)$ is continuous and satisfies the Lipschitz condition:

$$|f(t,u) - f(t,v)| = |-a(u-v)| \leq a|u-v|.$$

Thus, $L = a$. The constant M from Theorem 3.2 is given by $M = \frac{aT^\nu}{\Gamma(\nu+1)}$. In order to use the existence and stability results, we must have the following condition satisfied:$M + k\lambda < 1$. We have freedom of choice of the auxiliary operator g and the parameters k, λ since they are given. For instance, if $a = 0.5$, $T = 1$, and $\nu = 0.8$, then $\Gamma(1.8) \approx 0.931$, and $M \approx \frac{0.5\times 1}{0.931} \approx 0.537$. If we choose $\lambda = 0.1$ and $k = 1$, then $M + k\lambda = 0.537 + 0.1 = 0.637 < 1$. Hence, the condition (3.5) holds true.

The fractional relaxation-oscillation model (5.1) has by Theorem 3.2 a unique solution in $C[0,1]$. Moreover, Theorem 4.1 shows that the model is Ulam–Hyers stable. Let us calculate the stability constant explicitly. Following the proof of Theorem 3.2, we choose $\beta_0 = \frac{1-0.637}{2(1+M)} = \frac{0.363}{2(1.537)} \approx 0.118$. Then $\alpha_0 = 0.637 + \frac{0.363}{2} = 0.8185$. The constant $\delta = \frac{T^\nu}{\Gamma(\nu+1)} = \frac{1}{0.931} \approx 1.074$. As such, the stability constant is

$$C = \frac{(1+\beta_0)\delta}{1-\alpha_0} = \frac{(1.118)(1.074)}{1-0.8185} = \frac{1.2007}{0.1815} \approx 6.616.$$

The deviation in the solution from the perturbation of the system by an error ε will be of the order of maximum of 6.616 ε. This direct bound is really useful for engineers who design circuits that consist of fractional components.

The illustrative curves are shown in Figure 1.

5.2. Fractional logistic population model

The logistic population model is a fundamental model in population dynamics. To incorporate memory effects into the population growth process, we consider the following fractional logistic initial value problem:

$$\begin{cases} {}^C D^\nu u(t) = ru(t)\left(1 - \dfrac{u(t)}{K}\right), & t \in J = [0,T], \quad 0 < \nu < 1, \\ u(0) = u_0, \end{cases} \tag{5.2}$$

where $r > 0$ is the intrinsic growth rate, $K > 0$ is the carrying capacity, and $0 \leq u_0 \leq K$.

The corresponding nonlinear function is

$$f(t,u) = ru\left(1 - \frac{u}{K}\right).$$

Since f contains a quadratic term, it is not globally Lipschitz on $\mathbb{R}$. Therefore, Theorem 3.4 cannot be applied directly on the whole space $C(J,\mathbb{R})$ by using a global Lipschitz constant. Instead, we work on a suitable closed invariant subset of $C(J,\mathbb{R})$.

Define

$$\mathcal{X}_K = \{u \in C(J,\mathbb{R}) : 0 \leq u(t) \leq K, \quad t \in J\}.$$

Clearly, $\mathcal{X}_K$ is a nonempty closed subset of $C(J,\mathbb{R})$ endowed with the supremum norm

$$\|u\|_\infty = \max_{t\in J} |u(t)|.$$

For $u \in \mathcal{X}_K$, we have

$$0 \leq u(t) \leq K$$

and consequently

$$0 \leq f(t,u(t)) = ru(t)\left(1 - \frac{u(t)}{K}\right).$$

Moreover, the function

$$\phi(s) = rs\left(1 - \frac{s}{K}\right), \qquad 0 \leq s \leq K,$$

attains its maximum at $s = K/2$. Hence,

$$0 \leq f(t,u(t)) \leq \frac{rK}{4}. \tag{5.3}$$

The fractional differential equation (5.2) is equivalent to the Volterra integral equation

$$u(t) = u_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} f(s,u(s))\, ds. \tag{5.4}$$

Accordingly, define the integral operator

$$(\mathcal{T}u)(t) = u_0 + \frac{1}{\Gamma(\nu)} \int_0^t (t-s)^{\nu-1} ru(s)\left(1 - \frac{u(s)}{K}\right) ds.$$

We first verify the invariance of $\mathcal{X}_K$. For every $u \in \mathcal{X}_K$, by (5.3),

$$(\mathcal{T}u)(t) \geq u_0 \geq 0.$$

Furthermore,

$$\begin{aligned}(\mathcal{T}u)(t) &\leq u_0 + \frac{rK}{4\Gamma(\nu)} \int_0^t (t-s)^{\nu-1}\, ds \\ &= u_0 + \frac{rK}{4}\frac{t^\nu}{\Gamma(\nu+1)} \\ &\leq u_0 + \frac{rK}{4}\frac{T^\nu}{\Gamma(\nu+1)}.\end{aligned}$$

Therefore, if

$$u_0 + \frac{rK}{4}\frac{T^\nu}{\Gamma(\nu+1)} \leq K, \tag{5.5}$$

then

$$0 \leq (\mathcal{T}u)(t) \leq K, \qquad t \in J,$$

and hence

$$\mathcal{T}(\mathcal{X}_K) \subseteq \mathcal{X}_K.$$

Thus, $\mathcal{X}_K$ is an invariant closed subset for the integral operator.

Next, for $u, v \in \mathcal{X}_K$, we have

$$\begin{aligned}|f(t,u(t)) - f(t,v(t))| &= r\left|u(t) - v(t) - \frac{u(t)^2 - v(t)^2}{K}\right| \\ &= r|u(t)-v(t)|\left|1 - \frac{u(t)+v(t)}{K}\right| \\ &\leq r|u(t)-v(t)|,\end{aligned}$$

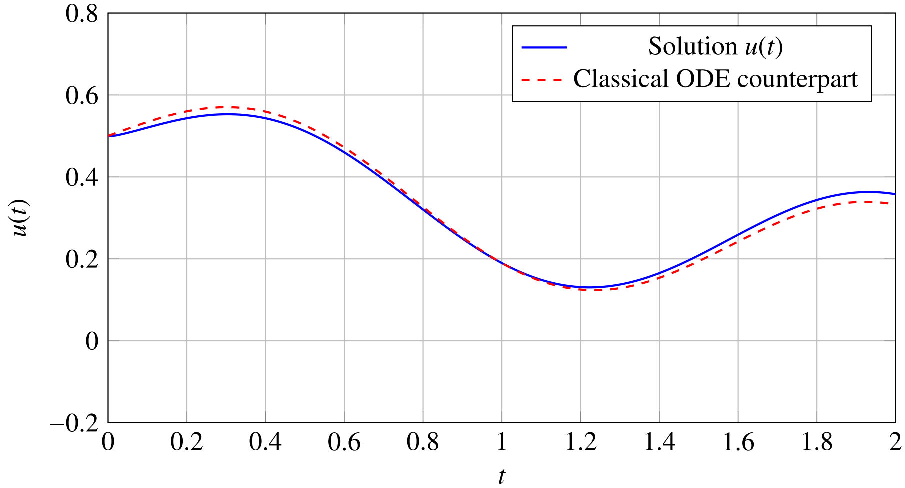

Figure 1. evComparison of the fractional relaxation–oscillation curve (blue solid) and its classical ordinary differential equation counterpart (red dashed). The fractional curve exhibits slower decay and a phase shift.

because $0 \le u(t), v(t) \le K$ implies

$$\left|1 - \frac{u(t)+v(t)}{K}\right| \le 1.$$

Consequently,

$$|f(t,u(t)) - f(t,v(t))| \le r|u(t)-v(t)|. \tag{5.6}$$

Thus, f is Lipschitz on the invariant interval $[0, K]$ with Lipschitz constant

$$L = r.$$

It follows that

$$\|\mathcal{T}u - \mathcal{T}v\|_\infty \le \frac{rT^\nu}{\Gamma(\nu+1)}\|u-v\|_\infty.$$

Hence, if

$$\frac{rT^\nu}{\Gamma(\nu+1)} < 1, \tag{5.7}$$

then the integral operator is contractive on $\mathcal{X}_K$. More generally, under the assumptions of Theorem 3.4, if the corresponding enriched contraction condition is satisfied on $\mathcal{X}_K$, Theorem 3.4 can be applied to obtain a unique fixed point of $\mathcal{T}$ in $\mathcal{X}_K$.

Therefore, under conditions (5.5) and (5.7), the fractional logistic problem (5.2) possesses a unique solution

$$u^* \in \mathcal{X}_K.$$

Furthermore, the Ulam–Hyers stability estimate established in Theorem 4.1 applies on this invariant subset. In particular, every approximate solution $u \in \mathcal{X}_K$ satisfying the corresponding perturbed fractional equation admits the estimate

$$\|u-u^*\|_\infty \le C_{\mathrm{bd}}\varepsilon,$$

where C_{bd} is the explicit upper bound obtained in Theorem 4.1.

This argument shows explicitly why the boundedness restriction in the logistic model is compatible with the fixed-point framework: the nonlinearity is Lipschitz on the invariant interval $[0, K]$, and the associated integral operator preserves the closed subset $\mathcal{X}_K$.

5.2.1. Parameter verification

For illustration, consider

$$r = 2, \qquad K = 10, \qquad u_0 = 0.5, \qquad \nu = 0.8, \qquad T = 0.2.$$

The invariance condition (5.5) gives

$$u_0 + \frac{rK}{4}\frac{T^\nu}{\Gamma(\nu+1)} = 0.5 + 5\frac{(0.2)^{0.8}}{\Gamma(1.8)} \approx 3.47 < 10 = K.$$

Hence,

$$\mathcal{T}(\mathcal{X}_K) \subseteq \mathcal{X}_K.$$

Furthermore,

$$\frac{rT^\nu}{\Gamma(\nu+1)} = \frac{2(0.2)^{0.8}}{\Gamma(1.8)} \approx 0.594 < 1.$$

Therefore, the integral operator is contractive on $\mathcal{X}_K$. Consequently, the hypotheses required for the application of the fixed-point theorem are satisfied on the invariant subset $\mathcal{X}_K$, and the fractional logistic problem admits a unique solution in this set. The Ulam–Hyers stability estimate follows from Theorem 4.1.

The corresponding growth curves are shown in Figure 2.

The sufficient existence conditions and stability bounds used for comparison are summarized in Table 1.

6. Illustrative examples and numerical simulations

We give a set of well-made examples to further substantiate our theoretical results and to demonstrate the sharpness of the conditions. These examples cover simple linear examples, more complex nonlinear examples.

Example 6.1 (A simple linear case)**.** Consider the fractional differential equation

$$^{C}D^{0.5}u(t) = \frac{1}{4}u(t), \quad t \in [0,1], \quad u(0) = 1.$$

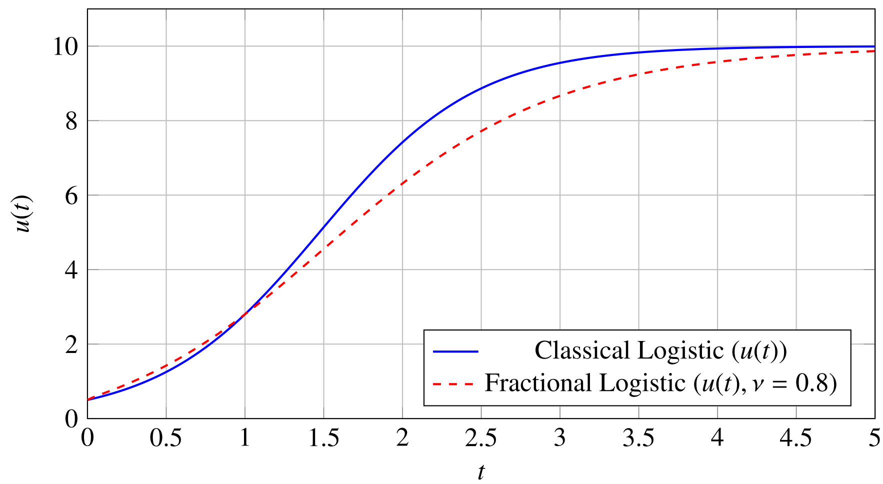

Figure 2. evComparison of classical logistic growth (blue solid) and fractional logistic growth with $u = 0.8$ (red dashed). The fractional curve exhibits more gradual initial growth and a slower transition to the carrying capacity, reflecting the modeled memory effect.

Table 1. Comparison of sufficient existence conditions.

Method	Sufficient condition	Stability bound
Banach contraction	$M < 1$	Not explicitly derived
Krasnosel'skii type	$\dfrac{LT^{\nu}}{\Gamma(\nu+1)} < \dfrac{1}{2}$	Not guaranteed
Proposed framework	$M + k\lambda < 1$	$C_{\mathrm{bd}} = \dfrac{(1+\beta_0)\delta}{1-\alpha_0}$

Here, $\nu = 0.5$, $T = 1$, $f(t,u) = \frac{1}{4}u$. The Lipschitz constant is $L = 0.25$. We compute $M = \frac{0.25\times 1^{0.5}}{\Gamma(1.5)} = \frac{0.25}{0.8862} \approx 0.2821$. We choose the auxiliary operator $gu = 0.1u$, so $\lambda = 0.1$, and set $k = 1$. Then $M + k\lambda = 0.2821 + 0.1 = 0.3821 < 1$. Condition (3.5) is satisfied. According to Theorem 3.2, a unique solution exists. We can verify this by noting that the exact solution is given by the Mittag-Leffler function:

$$u(t) = E_{0.5}\left(\frac{1}{4}t^{0.5}\right),$$

which is indeed a well-defined continuous function. The Ulam–Hyers stability constant can be calculated as follows: choose $\beta_0 = \frac{1-0.3821}{2(1+0.2821)} = \frac{0.6179}{2.5642} \approx 0.241$. Then $\alpha_0 = 0.3821 + \frac{0.6179}{2} = 0.691$. $\delta = \frac{1^{0.5}}{\Gamma(1.5)} \approx 1.128$. Thus, $C = \frac{(1+0.241)(1.128)}{1-0.691} = \frac{1.399}{0.309} \approx 4.53$. Consequently, if an approximate solution u obtained by a numerical method is independently verified to satisfy

$$\left\|{}^{C}D^{\nu}u(t) - f(t,u(t))\right\|_{\infty} \leq \varepsilon,$$

then Theorem 4.1 guarantees that

$$\|u - u^{*}\|_{\infty} \leq 4.53\,\varepsilon.$$

Here, ε represents the residual in the defining Ulam–Hyers inequality and should not be identified automatically with the local truncation error of the numerical method.

Example 6.2 (A nonlinear case)**.** Let us examine a more challenging nonlinear problem:

$${}^{C}D^{0.7}u(t) = \frac{1}{5}\sin(u(t)), \quad t \in [0,2], \quad u(0) = 0.$$

Here, $L = \frac{1}{5} = 0.2$. We have $\nu = 0.7, T = 2$. $\Gamma(1.7) \approx 0.908$. So $M = \frac{0.2\times 2^{0.7}}{0.908} = \frac{0.2\times 1.6245}{0.908} \approx 0.3579$. We choose $gu = 0.05u$ ($\lambda = 0.05$) and $k = 1.5$. Then $M + k\lambda = 0.3579 + 1.5(0.05) = 0.3579 + 0.075 = 0.4329 < 1$. Existence and uniqueness are guaranteed. To illustrate stability, let us compute the constant. $\delta = \frac{2^{0.7}}{0.908} \approx 1.789$. Let $\beta_0 = \frac{1-0.4329}{2(1+0.3579)} = \frac{0.5671}{2.7158} \approx 0.2088$. $\alpha_0 = 0.4329 + \frac{0.5671}{2} = 0.71645$. Then $C = \frac{(1+0.2088)(1.789)}{1-0.71645} = \frac{2.161}{0.28355} \approx 7.62$. This example demonstrates that our method works effectively for nonlinear equations, provided the Lipschitz constant is not too large.

Example 6.3 (Failure of the sufficient contraction condition)**.** Consider the linear fractional differential equation

$${}^{C}D^{0.2}u(t) = 5u(t), \qquad t \in [0,1], \qquad u(0) = 1.$$

Here,

$$\nu = 0.2, \qquad T = 1, \qquad L = 5.$$

Therefore,

$$M = \frac{LT^{\nu}}{\Gamma(\nu+1)} = \frac{5}{\Gamma(1.2)} \approx 5.445 > 1.$$

Even with the choice $g = 0$ and $k = 1$, condition (3.5),

$$M + k\lambda < 1,$$

cannot be satisfied, since $\lambda = 0$ gives

$$M + k\lambda = M \approx 5.445 > 1.$$

Hence, the sufficient contraction condition of Theorem 3.4 is not satisfied, and consequently Theorem 3.4 does not provide an existence or uniqueness conclusion for this problem.

However, failure of the sufficient contraction condition does not imply nonexistence, nonuniqueness, or blow-up of solutions. In fact, this linear Caputo fractional differential equation has the explicit Mittag–Leffler solution

$$u(t) = E_{0.2}\Big(5t^{0.2}\Big),$$

where

$$E_\nu(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\nu n + 1)}$$

denotes the Mittag–Leffler function. Thus, a solution exists and is well defined on the interval $[0, 1]$.

This example therefore illustrates only that the sufficient contraction criterion $M + k\lambda < 1$ is inconclusive when its hypotheses are not satisfied. It should not be interpreted as evidence of nonexistence, nonuniqueness, blow-up, or sharpness of condition (3.5).

Example 6.4 (Numerical illustration of the stability estimate)**.** We consider the fractional differential equation studied in Example 6.1 and illustrate the Ulam–Hyers stability estimate obtained in Theorem 4.1. For the parameters used in Example 6.1, the explicit upper bound for the Ulam–Hyers stability constant is

$$C_{\mathrm{bd}} = \frac{(1+\beta_0)\delta}{1-\alpha_0} \approx 4.53, \qquad \delta = \frac{T^\nu}{\Gamma(\nu+1)}.$$

Let u be an approximate solution satisfying the residual condition

$$\left\|{}^C D^\nu u(t) - f(t, u(t))\right\|_\infty \le \varepsilon, \qquad t \in [0, T].$$

Then, by Theorem 4.1, the unique exact solution u^* satisfies

$$\|u - u^*\|_\infty \le C_{\mathrm{bd}}\varepsilon \approx 4.53\,\varepsilon.$$

Thus, for example, if an independently computed approximate solution is verified to satisfy the residual inequality with $\varepsilon = 0.01$, then the theoretical stability estimate gives

$$\|u - u^*\|_\infty \le 4.53(0.01) = 0.0453.$$

This value is a theoretical upper bound furnished by the Ulam–Hyers stability result. It should not be interpreted as the actual numerical error, nor does it imply that the bound is sharp. A numerical verification would require specifying the numerical scheme, discretization parameters, perturbation realization, and computing the residual explicitly.

7. Discussion and comparison with existing literature

The results presented in this paper have several advantages over existing approaches in the literature. First, the enriched (g, α, β, k)-functional contraction is introduced as a unified framework which includes several classical fixed-point theorems as special cases. In contrast to the Banach contractions, which demand a fixed contraction coefficient $\lambda < 1$, our method is flexible enough to have the functional control coefficients, $\alpha(t)$ and $\beta(t)$, change with the distance between points. This is especially useful for fractional order differential equations where the integral operator may have non-uniform contractive properties.

Second, our explicit Ulam–Hyers stability constant is a substantial extension of many existing ones. In the study of Abbas *et al.* [15] stability constants are, for instance, not given by an explicit formula but in the abstract existential bounds. Our Theorem gives a simple computable formula for C as a function of the parameters $\alpha_0, \beta_0, \delta$ that immediately relate to the problem data. This allows our results to be readily utilized in the numerical analysis and in actual engineering applications.

Third, the auxiliary operator g provides an additional structural component of the proposed contractive framework. For the particular choice $g = \lambda I$ considered in Theorem 3.4, the sufficient condition becomes

$$M + k\lambda < 1.$$

Since $k\lambda \ge 0$, this condition necessarily implies $M < 1$. Thus, in this particular realization, the auxiliary operator does not enlarge the admissible range of the Lipschitz parameter beyond the classical Banach condition. Its role is instead to incorporate an operator-dependent term into the contractive inequality and to provide a unified framework together with the functional control functions α and β. The investigation of alternative choices of g that may lead to genuinely different admissibility conditions is left for future work.

Our work differs from that of Baleanu *et al.* [32] which investigated the existence of solutions of fractional differential equations for different fixed-point theorems, in that we consider both existence of solutions and stability at the same time. A few authors have investigated the Ulam–Hyers stability of fractional differential equations, but generally with the condition that the nonlinearity is globally Lipschitz and the operator is a standard contraction. Our enrichment method gets rid of such requirements and offers a stronger stability guarantee.

The comparison in Table 1 should be interpreted in terms of the structure of the sufficient conditions and the availability of an explicit Ulam–Hyers stability bound. In particular, the condition $M + k\lambda < 1$ does not provide a larger admissible region than $M < 1$, since $k\lambda \ge 0$. The principal advantage claimed here is therefore the unified contractive formulation and the explicit stability estimate, rather than an enlargement of the Lipschitz parameter range.

8. Conclusion

In this paper, we have investigated the existence, uniqueness, and Ulam–Hyers stability of solutions for a class of nonlinear fractional differential equations. The analysis is based on a class of mappings termed enriched (g, α, β, k)-functional contractions, which incorporates functional control parameters and an auxiliary operator into the metric setting. The precise relationship of this framework to the classical Banach, Kannan, and

Chatterjea contraction classes is retained for author verification in the Authors' Queries.

Under the stated conditions, we proved convergence of the Picard iterates and existence of a unique fixed point, and then applied the abstract result to a Caputo-type fractional differential equation. The differential equation was recast as an equivalent integral equation, and sufficient conditions for existence and uniqueness of solutions were obtained. For the fractional differential equation considered here, we obtained the sufficient condition

$$M + k\lambda < 1.$$

Since $k\lambda \geq 0$, this condition implies the classical requirement $M < 1$. Thus, we do not claim that this particular condition enlarges the classical admissible parameter region. Instead, the contribution of the proposed framework lies in the unified incorporation of the auxiliary operator g, the enrichment parameter k, and the functional control functions α and β, together with the corresponding fixed-point and Ulam–Hyers stability analysis.

Significantly, we have obtained an explicit upper bound for the Ulam–Hyers stability constant in our stability analysis. The estimate

$$C_{\mathrm{bd}} = \frac{(1+\beta_0)\delta}{1-\alpha_0}$$

is directly computable from the problem parameters and provides a quantitative measure of the deviation of an approximate solution from the corresponding exact solution. The present analysis establishes the validity of this upper bound but does not prove that it is optimal or sharp.

We illustrated the theoretical results through applications to fractional relaxation–oscillation and fractional logistic population models. The examples and numerical simulation demonstrate the applicability of the derived sufficient conditions and provide numerical support for the explicit Ulam–Hyers stability bound. The numerical results do not establish the optimality or sharpness of the stability bound.

There are several promising directions for future research. First, one could extend our analysis to systems of fractional differential equations or to fractional partial differential equations. The enriched contraction framework is flexible enough to be adapted to higher-dimensional settings. Second, it would be interesting to explore the Hyers–Ulam–Rassias stability, which allows for a more general perturbation function, within the context of our enriched contractions. Third, the role of the auxiliary operator g could be further investigated; for instance, choosing g as a fractional integral operator itself might yield even sharper results. We hope that the unified framework presented here will inspire further developments in the qualitative theory of fractional differential equations and stimulate new applications in the mathematical modeling of complex systems.

Data availability

The data supporting the findings of this study are available from the corresponding author upon reasonable request.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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