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Soft S_p-compactness and soft S_p-Lindelöfness: characterizations, relationships, and preservation theorems

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Abstract

Two novel classes of soft topological spaces, soft S_p-compact spaces and soft S_p-Lindelöf spaces, are introduced through soft S_p-open covers. Several equivalent characterizations are proved, including one via families of soft S_p-closed sets satisfying the finite (respectively, countable) soft intersection property. The relationships of the introduced notions with the principal covering-type properties in soft topology are investigated, and the exact structural conditions, soft local indiscreteness, the soft T_1 axiom, and soft extremal disconnectedness, under which such relationships hold as equivalences or one-way implications are determined. Soft S_p-compact sets, soft S_p-Lindelöf sets, and their subspace analogues are also introduced, and their behaviour under soft clopen and soft S_p-closed subsets is studied. Several preservation theorems under various types of soft continuity- and openness-type functions are proved. Suitable counterexamples are provided to demonstrate that the converses of the main implication relations do not hold in general.

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1. Introduction and preliminaries

The theory of soft sets, initiated by Molodtsov [1] in 1999, was proposed as a parameterized framework for modelling uncertainty. Unlike fuzzy set theory, rough set theory, and interval mathematics, in which the absence of any adequate parametrization tool creates persistent difficulties, the soft-set framework provides such a tool intrinsically. Building on Molodtsov's work, the algebraic side of the theory was developed extensively by Maji *et al.* [2], who introduced the fundamental soft-set operations. Over the years, various extensions of soft set theory have emerged to address complex real-world applications: fuzzy soft set-valued maps were introduced to establish fixed-point theorems in homotopy [3], bipolar soft neighborhood structures were formulated for decision-making applications [4], and complex triple-valued neutrosophic soft topological frameworks were developed for artificial intelligence (AI)-driven signal analysis [5].

Soft topology, in the sense that is now standard, was inaugurated by Shabir and Naz [6], who transported the classical Kuratowski framework into the soft setting and introduced soft topological spaces, soft open and soft closed sets, soft closure, and soft interior. Their foundational paper triggered a rapid expansion of the field, and a substantial body of work has since been devoted to reformulating and extending classical topolog-

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ical notions, including separation axioms, continuity, connectedness, and covering properties, within this parameterized environment [7–12].

A recurring and particularly fruitful theme in this program has been the study of generalized soft open sets. Motivated by their crisp counterparts (semi-open, pre-open, α-open, and regular open sets), several such classes have been introduced and analysed in the soft setting: soft semi-open sets by Chen [13], soft pre-open and soft regular open sets by Ilango and Ravindran [14], soft α-open sets by Akdag and Ozkan [15], and soft θ-open sets by Georgiou, Megaritis, and Petropoulos [11]. Each new class of soft open sets induces, in a natural way, its own notion of continuity, its own separation axioms, and, of central importance to the present paper, its own covering-type properties.

Covering properties occupy a distinguished position in general topology, and their soft-theoretic analogues have accordingly attracted considerable attention. Beyond soft compactness and soft Lindelöfness themselves [7], [10], the literature now contains a rich hierarchy of variants: soft semi-compactness and soft semi-Lindelöfness [16], [17]; soft pre-compactness and soft pre-Lindelöfness [18]; soft α-compactness and soft α-Lindelöfness [19]; mildly soft compactness and its Lindelöf counterpart [20]; almost soft compactness [20]; soft nearly compactness [21]; and soft$_{\text{int}}$ compactness introduced via soft regular closed sets [22]. Furthermore, compactness, connectedness, and local compactness have also been successfully investigated using other generalized classes, such as soft somewhat open sets [23] and soft δ-open sets [24]. Each of these notions captures a different degree of finiteness of covers and interacts in a subtle way with the corresponding classes of generalized soft open sets.

Very recently, in a companion paper [25], the present authors introduced the class of soft S_p-open sets, a class that occupies a strictly intermediate position between soft open and soft semi-open sets: every soft S_p-open set is soft semi-open, but not conversely. The basic structural properties of this class, together with the associated soft S_p-continuous, soft S_p-irresolute, and soft S_p-open functions, have been investigated in [26–28]. Nevertheless, no covering-type theory has thus far been developed for soft S_p-open sets, and the natural question of what form soft compactness and soft Lindelöfness assume when the covering family is drawn from $\widetilde{S}S_pO(\widetilde{X})$ has remained open.

The present paper addresses this question. Our aim is to introduce and to study systematically two new covering-type properties, which we call soft S_p-compactness and soft S_p-Lindelöfness, and to locate them precisely within the existing hierarchy of soft covering notions. The main contributions of the paper are the following. First, we introduce soft S_p-compact and soft S_p-Lindelöf spaces and establish several equivalent characterizations, including one formulated in terms of families of soft S_p-closed sets enjoying the finite (respectively, countable) soft intersection property. Second, we conduct a detailed comparative analysis between the new notions and the existing ones (soft compactness, soft semi-compactness, pre-compactness, and α-compactness, soft nearly compactness, mildly soft compactness, almost soft compactness, and soft$_{\text{int}}$ compactness), and identify precise structural hypotheses (soft local indiscreteness, the soft T_1 axiom, soft extremal disconnectedness, and the soft p-Alexandroff property) under which these notions coincide or one implies another.

Third, we develop the corresponding local theory by introducing soft S_p-compact sets, soft S_p-Lindelöf sets, and their subspace analogues, and by examining their behaviour under soft clopen and soft S_p-closed subsets, as well as under finite soft unions and intersections. Fourth, we investigate the invariance of soft S_p-compactness and soft S_p-Lindelöfness under a variety of soft mappings, including soft S_p-continuous, soft S_p-irresolute, soft almost S_p-continuous, soft weakly S_p-continuous, and soft S_p-open functions. Throughout the paper, illustrative examples and counterexamples are provided to demonstrate that the notions introduced here are proper generalizations of the existing ones and to clarify the non-reversibility of the established implications between soft S_p-compactness and related soft covering properties.

Throughout this work, let $\mathcal{E}$ denote a set of parameters and let $\mathcal{X}$ be a fixed non-empty set. By a soft set over $\mathcal{X}$ relative to $\mathcal{E}$ we mean a pair $(\mathcal{M},\ \mathcal{E}) = \{(\varepsilon, \mathcal{M}(\varepsilon)) :\ \varepsilon\in \mathcal{E},\ \mathcal{M}(\varepsilon)\in \mathcal{P}(\mathcal{X})\}$ determined by a function $\mathcal{M} : \mathcal{E} \to \mathcal{P}(\mathcal{X})$, and the collection of all such soft sets is written $\widetilde{S}S(\mathcal{X},\mathcal{E})$. For simplicity, the pair $(\mathcal{X},\ \mathcal{E})$ is denoted as $\widetilde{\mathcal{X}}$. According to Maji *et al.* [2], the null soft set, expressed as $\widetilde{\emptyset}$, occurs when $\mathcal{M}(\varepsilon) = \emptyset,\ \forall\ \varepsilon\in \mathcal{E}$. Conversely, the absolute soft set, expressed as $\widetilde{\mathcal{X}}$, is formed when $\mathcal{M}(\varepsilon)= \mathcal{X},\ \forall\ \varepsilon\in \mathcal{E}$. Furthermore, the soft complement of a soft set $(\mathcal{M},\ \mathcal{E})$ is represented by $\widetilde{\mathcal{X}}\backslash(\mathcal{M},\ \mathcal{E}) = (\mathcal{M},\ \mathcal{E})^{c} = (\mathcal{M}^{c},\mathcal{E})$, relying on the mapping $\mathcal{M}^{c}\colon \mathcal{E} \to \mathcal{P}(\mathcal{X})$ which is defined by $\mathcal{M}^{c}(\varepsilon)= \mathcal{X} - \mathcal{M}(\varepsilon),\ \forall\ \varepsilon\in \mathcal{E}$.

As outlined in Ref. [7], for a collection of soft sets $(\mathcal{M}_{\mathrm{i}},\ \mathcal{E})\widetilde{\in}\widetilde{S}S(\mathcal{X},\mathcal{E})$, where $\mathrm{i}\in \mathcal{I}$, their soft union produces a soft set $(\mathcal{M},\ \mathcal{E})\widetilde{\in}\widetilde{S}S(\mathcal{X},\mathcal{E})$ such that $\mathcal{M}(\varepsilon) = \widetilde{\cup}_{\mathrm{i}\in\mathcal{I}}\mathcal{M}_{\mathrm{i}}(\varepsilon)$ for all $\varepsilon\in \mathcal{E}$. Similarly, their soft intersection yields $(\mathcal{M},\ \mathcal{E})\widetilde{\in}\widetilde{S}S(\mathcal{X},\mathcal{E})$ where $\mathcal{M}(\varepsilon) = \widetilde{\cap}_{\mathrm{i}\in\mathcal{I}}\mathcal{M}_{\mathrm{i}}(\varepsilon)$ for all $\varepsilon\in \mathcal{E}$. A soft point [29] [7], denoted by $\widetilde{\varepsilon_x}$, is a specific soft set $(\mathcal{M},\ \mathcal{E})$ where $\mathcal{M}(\varepsilon) = \{x\}$ and $\mathcal{M}(\acute{\varepsilon}) = \emptyset$ for all $\acute{\varepsilon}\in \mathcal{E} - \{\varepsilon\}$. We state that $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{N},\ \mathcal{E})$ provided $\varepsilon\in \mathcal{E}$ and $\{x\} \subseteq \mathcal{N}(\varepsilon)$. The entire class of soft points over $\mathcal{X}$ is designated by $\widetilde{S}P(\widetilde{\mathcal{X}})$. Let $\widetilde{S}S(\mathcal{X},\mathcal{E})$ and $\widetilde{S}S(Y,\acute{\mathcal{E}})$ be two collections of soft sets, over $\mathcal{X}$ and Y with parameter sets $\mathcal{E}$ and $\acute{\mathcal{E}}$, respectively, and let $u :\ \mathcal{X} \to Y$ and $p :\ \mathcal{E} \to \acute{\mathcal{E}}$ be two functions. A soft function $\widetilde{f}_{pu} :\ \widetilde{S}S(\mathcal{X},\mathcal{E}) \to \widetilde{S}S(Y,\acute{\mathcal{E}})$ is defined as follows:

- The soft image of $(\mathcal{M},\ \mathcal{E})\widetilde{\in}\widetilde{S}S(\mathcal{X},\mathcal{E})$ under $\widetilde{f}_{pu}$ is the soft set $(\widetilde{f}_{pu}(\mathcal{M}),\acute{\mathcal{E}})\widetilde{\in}\widetilde{S}S(Y,\acute{\mathcal{E}}),\ \forall\ \beta\in\acute{\mathcal{E}}\ \widetilde{f}_{pu}(\mathcal{M})(\beta) = \begin{cases} u(\bigcup_{\alpha\in p^{-1}(\beta)\cap\mathcal{E}}\ \mathcal{M}(\alpha)), & if\ p^{-1}(\beta)\cap\mathcal{E}\neq\emptyset \\ \widetilde{\emptyset}, & if\ p^{-1}(\beta)\cap\mathcal{E} = \emptyset \end{cases}$.
- The soft inverse image of $(\mathcal{N},\acute{\mathcal{E}})\widetilde{\in}\widetilde{S}S(Y,\acute{\mathcal{E}})$, under $\widetilde{f}_{pu}$ is the soft set $(\widetilde{f}^{-1}_{pu}(\mathcal{N}),\mathcal{E})\widetilde{\in}\widetilde{S}S(\mathcal{X},\mathcal{E}),\ \forall\ \alpha\in\mathcal{E}$: $\widetilde{f}^{-1}_{pu}(\mathcal{N})(\alpha) = u^{-1}(\mathcal{N}(p(\alpha)))$ [30].

Furthermore, the soft function $\widetilde{f}_{pu}$ is classified as soft injective, soft surjective, or soft bijective provided that both un-

derlying classical functions u and p are injective, surjective, or bijective, respectively [31].

The concept of a soft topological space ($\widetilde{S}TS$) over $\mathcal{X}$ was established in Ref. [6] as a triplet $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ (often abbreviated to just $\widetilde{\mathcal{X}}$). In this context, the family $\widetilde{\mathcal{T}} \widetilde{\subseteq} \widetilde{S}S(\mathcal{X}, \mathcal{E})$, called a "soft topology" on $\widetilde{\mathcal{X}}$, must contain $\widetilde{\varnothing}$ and $\widetilde{\mathcal{X}}$, and it must satisfy closure under arbitrary soft unions and finite soft intersections. Elements belonging to $\widetilde{\mathcal{T}}$ are classified as soft open sets, while their corresponding soft complements (elements of $\widetilde{\mathcal{T}}^c$) are recognized as soft closed sets [9]. For any subset $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$, its soft closure, $\widetilde{scl}(\mathcal{M}, \mathcal{E})$, is formulated as $\widetilde{\cap}\{(C, \mathcal{E}) : (C, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}^c and\ (\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} (C, \mathcal{E})\}$. This represents the minimal soft closed set encompassing $(\mathcal{M}, \mathcal{E})$ [6].

Conversely, the soft interior, $\widetilde{sint}(\mathcal{M}, \mathcal{E})$, is defined as $\widetilde{\cup}\{(M, \mathcal{E}) : (M, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}\ and\ (M, \mathcal{E}) \widetilde{\subseteq} (\mathcal{M}, \mathcal{E})\}$, which makes it the maximal soft open set contained within $(\mathcal{M}, \mathcal{E})$ [9]. Finally, a soft subspace of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is described by the triple $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$, where $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} \widetilde{\mathcal{X}}$. The collection $\widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})} = \{(\mathcal{M}, \mathcal{E}) \widetilde{\cap} (\mathcal{O}, \mathcal{E}) ; (\mathcal{O}, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}\}$ constitutes the "soft relative topology" on $(\mathcal{M}, \mathcal{E})$ [6].

Before proceeding to the main results in the following sections, we outline the requisite foundational definitions for the reader's convenience.

Definition 1.1. Let $(\mathcal{M}, \mathcal{E})$ be a soft subset of a $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$. We classify $(\mathcal{M}, \mathcal{E})$ as a soft semi-open [13], soft pre-open [14], soft α-open [15], or soft regular open [14] set provided that it satisfies the respective condition: $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} \widetilde{scl}(\widetilde{sint}(\mathcal{M}, \mathcal{E}))$, $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} \widetilde{sint}(\widetilde{scl}(\mathcal{M}, \mathcal{E}))$, $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} \widetilde{sint}(\widetilde{scl}(\widetilde{sint}(\mathcal{M}, \mathcal{E})))$, or $(\mathcal{M}, \mathcal{E}) = \widetilde{sint}(\widetilde{scl}(\mathcal{M}, \mathcal{E})))$.

Consequently, $(\mathcal{M}, \mathcal{E})$ is termed a soft semi-closed [16], soft pre-closed [14], soft α-closed [15], or soft regular closed [32] set whenever its soft complement belongs to the corresponding class of soft open sets mentioned above.

The collections of all such soft open sets in $\widetilde{\mathcal{X}}$ are denoted by $\widetilde{S}SO(\widetilde{\mathcal{X}})$, $\widetilde{S}PO(\widetilde{\mathcal{X}})$, $\widetilde{S}\alpha O(\widetilde{\mathcal{X}})$, and $\widetilde{S}RO(\widetilde{\mathcal{X}})$, respectively. Similarly, their closed counterparts are designated $\widetilde{S}SC(\widetilde{\mathcal{X}})$, $\widetilde{S}PC(\widetilde{\mathcal{X}})$, $\widetilde{S}\alpha C(\widetilde{\mathcal{X}})$, and $\widetilde{S}RC(\widetilde{\mathcal{X}})$, respectively.

Definition 1.2. ([25], [26]) A soft subset $(\mathcal{M}, \mathcal{E})$ of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is designated as soft S_p-open, if $(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{S}SO(\widetilde{\mathcal{X}})$ and, for every $\widetilde{\varepsilon_x} \widetilde{\in} (\mathcal{M}, \mathcal{E})$, there exists $(B, \mathcal{E}) \widetilde{\in} \widetilde{S}PC(\widetilde{\mathcal{X}})$ such that $\widetilde{\varepsilon_x} \widetilde{\in} (B, \mathcal{E}) \widetilde{\subseteq} (\mathcal{M}, \mathcal{E})$. The soft complement of such a set referred to as soft S_p-closed. We denote the families comprising all soft S_p-open and soft S_p-closed subsets of $\widetilde{\mathcal{X}}$ by $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$ and $\widetilde{S}S_pC(\widetilde{\mathcal{X}})$, respectively.

To clarify this notion, we present an example of a soft S_p-open set that is not soft open.

Example 1.3. Let $\mathcal{X} = \{\varkappa_1, \varkappa_2, \varkappa_3\}$, and $\mathcal{E} = \{\varepsilon_1, \varepsilon_2\}$ with the soft topology $\widetilde{\mathcal{T}} = \{\widetilde{\emptyset}, \widetilde{\mathcal{X}}, (\mathcal{M}_1, \mathcal{E}), (\mathcal{M}_2, \mathcal{E}), (\mathcal{M}_3, \mathcal{E}), (\mathcal{M}_4, \mathcal{E}), (\mathcal{M}_5, \mathcal{E})\}$ on $\widetilde{\mathcal{X}}$, where $\widetilde{\emptyset} = \{(\varepsilon_1, \varnothing), (\varepsilon_2, \varnothing)\}$, $\widetilde{\mathcal{X}} = \{(\varepsilon_1, \mathcal{X}), (\varepsilon_2, \mathcal{X})\}$, $(\mathcal{M}_1, \mathcal{E}) = \{(\varepsilon_1, \{\varkappa_2, \varkappa_3\}), (\varepsilon_2, \{\varkappa_1, \varkappa_3\})\}$, $(\mathcal{M}_2, \mathcal{E}) = \{(\varepsilon_1, \{\varkappa_1, \varkappa_2\}), (\varepsilon_2, \{\varkappa_1, \varkappa_2\})\}$, $(\mathcal{M}_3, \mathcal{E}) = \{(\varepsilon_1, \{\varkappa_2\}), (\varepsilon_2, \{\varkappa_1\})\}$, $(\mathcal{M}_4, \mathcal{E}) = \{(\varepsilon_1, \mathcal{X}), (\varepsilon_2, \{\varkappa_1, \varkappa_3\})\}$, $(\mathcal{M}_5, \mathcal{E}) = \{(\varepsilon_1, \{\varkappa_1, \varkappa_2\}), (\varepsilon_2, \{\varkappa_1\})\}$. Thus, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $\widetilde{S}TS$ over $\mathcal{X}$. The soft set $(\mathcal{M}_6, \mathcal{E}) = \{(\varepsilon_1, \{\varkappa_2\}), (\varepsilon_2, \{\varkappa_1, \varkappa_2\})\} \widetilde{\in} \widetilde{S}S_pO(\widetilde{\mathcal{X}})$ but $(\mathcal{M}_6, \mathcal{E}) \widetilde{\notin} \widetilde{\mathcal{T}}$.

Definition 1.4. Let $(\mathcal{M}, \mathcal{E})$ be a soft subset of a $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$. We classify $(\mathcal{M}, \mathcal{E})$ as soft clopen [20] if it is simultaneously soft open and soft closed. Following the same principle, $(\mathcal{M}, \mathcal{E})$ is designated as soft semi-clopen [17] or soft α-clopen [19] provided that it is both soft semi-open and soft semi-closed, or both soft α-open and soft α-closed, respectively. The collections of all such sets in $\widetilde{\mathcal{X}}$ by $\widetilde{S}CO(\widetilde{\mathcal{X}})$, $\widetilde{S}SCO(\widetilde{\mathcal{X}})$, and $\widetilde{S}\alpha CO(\widetilde{\mathcal{X}})$, respectively.

Definition 1.5. ([13], [14], [15], [16]) Let $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$. For each $\mathfrak{X} \in \{\text{semi, pre, } \alpha\}$, let $\delta \in \{S, P, \alpha\}$ denote its corresponding index such that: $\delta = \begin{cases} S, & if\ \mathfrak{X} = semi \\ P, & if\ \mathfrak{X} = pre \\ \alpha, & if\ \mathfrak{X} = \alpha \end{cases}$. Then,

1. the soft $\mathfrak{X}$-closure of $(\mathcal{M}, \mathcal{E})$ is $\widetilde{\cap}\{(C, \mathcal{E}) : (C, \mathcal{E}) \widetilde{\in} \widetilde{S}\delta C(\widetilde{\mathcal{X}}), (\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} (C, \mathcal{E})\}$, written $\widetilde{\mathfrak{s}}scl(\mathcal{M}, \mathcal{E})$, $\widetilde{\mathfrak{s}}pcl(\mathcal{M}, \mathcal{E})$, $\widetilde{\mathfrak{s}}\alpha cl(\mathcal{M}, \mathcal{E})$, respectively.
2. the soft $\mathfrak{X}$-interior of $(\mathcal{M}, \mathcal{E})$ is $\widetilde{\cup}\{(\mathcal{O}, \mathcal{E}) : (\mathcal{O}, \mathcal{E}) \widetilde{\in} \widetilde{S}\delta O(\widetilde{\mathcal{X}}), (\mathcal{O}, \mathcal{E}) \widetilde{\subseteq} (\mathcal{M}, \mathcal{E})\}$ written $\widetilde{\mathfrak{s}}sint(\mathcal{M}, \mathcal{E})$, $\widetilde{\mathfrak{s}}pint(\mathcal{M}, \mathcal{E})$, $\widetilde{\mathfrak{s}}\alpha int(\mathcal{M}, \mathcal{E})$, respectively.

Definition 1.6. [11] Let $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ and $\widetilde{\mathfrak{s}}\theta int(\mathcal{M}, \mathcal{E}) = \widetilde{\cup}\{(B, \mathcal{E}) ; \widetilde{scl}(B, \mathcal{E}) \widetilde{\subseteq} (\mathcal{M}, \mathcal{E}), (B, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}\}$. The soft set $(\mathcal{M}, \mathcal{E})$ is called soft θ-open if $\widetilde{\mathfrak{s}}\theta int(\mathcal{M}, \mathcal{E}) = (\mathcal{M}, \mathcal{E})$. Consequently, any soft set whose complement is soft θ-open is classified as a soft θ-closed set. The collections comprising all soft θ-open and soft θ-closed sets over $\widetilde{\mathcal{X}}$ are symbolized by $\widetilde{S}\theta O(\widetilde{\mathcal{X}})$ and $\widetilde{S}\theta C(\widetilde{\mathcal{X}})$, respectively.

Definition 1.7. A $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is categorized as follows:

1. Soft extremally disconnected $(soft_{E.\ D.})$ [33], if $\widetilde{scl}(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}$ for all $(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}$.
2. Soft locally indiscrete $(soft_{Lo.\ Ind.})$ [34], if $(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}^c$ for all $(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}$.
3. Soft p-Alexandroff $(soft_{p-Ale})$ [25], if any soft intersection of soft pre-open sets is soft pre-open. Or equivalently any soft union of soft pre-closed sets is soft pre-closed.
4. Soft T_1-space $(soft_{T_1-space})$ [12], if $\widetilde{\varepsilon_x}$, $\widetilde{\varepsilon_y} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})$ with $\widetilde{\varepsilon_x} \neq \widetilde{\varepsilon_y}$, there is $(\mathcal{M}_1, \mathcal{E})$, $(\mathcal{M}_2, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}$ such that $\widetilde{\varepsilon_x} \widetilde{\in} (\mathcal{M}_1, \mathcal{E})$, $\widetilde{\varepsilon_y} \widetilde{\notin} (\mathcal{M}_1, \mathcal{E})$ and $\widetilde{\varepsilon_y} \widetilde{\in} (\mathcal{M}_2, \mathcal{E})$, $\widetilde{\varepsilon_x} \widetilde{\notin} (\mathcal{M}_2, \mathcal{E})$.
5. Soft S_p-T_2-space $(soft_{S_p-T_2-space})$ [28], if $\widetilde{\varepsilon_x}$, $\widetilde{\varepsilon_y} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})$ with $\widetilde{\varepsilon_x} \neq \widetilde{\varepsilon_y}$, there is $(\mathcal{M}_1, \mathcal{E}), (\mathcal{M}_2, \mathcal{E}) \widetilde{\in} \widetilde{S}S_pO(\widetilde{\mathcal{X}})$ such that $\widetilde{\varepsilon_x} \widetilde{\in} (\mathcal{M}_1, \mathcal{E})$, $\widetilde{\varepsilon_y} \widetilde{\in} (\mathcal{M}_2, \mathcal{E})$, and $(\mathcal{M}_1, \mathcal{E}) \widetilde{\cap} (\mathcal{M}_2, \mathcal{E}) = \widetilde{\varnothing}$.

Definition 1.8. A soft function $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ is considered to be soft S_p-continuous [27] at a soft point $\widetilde{\varepsilon_x} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})$, if for each $(B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{\sigma}$ containing $\widetilde{f_{pu}}(\widetilde{\varepsilon_x})$, there exists $(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{S}S_pO(\widetilde{\mathcal{X}})$ containing $\widetilde{\varepsilon_x}$ such that $\widetilde{f_{pu}}(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} (B, \acute{\mathcal{E}})$. Furthermore, if this property holds true at every single soft point $\widetilde{\varepsilon_x} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})$, the function $\widetilde{f_{pu}}$ is broadly termed a soft S_p-continuous function.

Definition 1.9. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{\mathcal{Y}}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be a soft function. We classify $\widetilde{f_{pu}}$ as;

1. soft continuous [7] if $\widetilde{f_{pu}^{-1}}(B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{\mathcal{T}}$, $\forall\ (B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{\sigma}$.
2. Soft S_p-irresolute [28] if $\widetilde{f_{pu}^{-1}}(B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{S} S_p O(\widetilde{\mathcal{X}})$, $\forall (B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{S} S_p O(\widetilde{\mathcal{Y}})$.
3. Soft irresolute [16] if $\widetilde{f_{pu}^{-1}}(B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{S} S O(\widetilde{\mathcal{X}})$, $\forall (B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{S} S O(\widetilde{\mathcal{Y}})$.
4. Soft irresolute open [35] if $\widetilde{f_{pu}}(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{S} S O(\widetilde{\mathcal{Y}})$, $\forall\ (\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{S} S O(\widetilde{\mathcal{X}})$.
5. Soft open [8] (resp., soft S_p-open [28]) if $\widetilde{f_{pu}}(\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{\sigma}$ (resp., $\widetilde{S} S_p O(\widetilde{\mathcal{Y}})$), $\forall\ (\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{\mathcal{T}}$.

Definition 1.10. [29] A soft set $(\mathcal{M},\mathcal{E})$ over $\mathcal{X}$ is known as a finite (resp., a countable) soft set if $\mathcal{M}(\varepsilon)$ is finite (resp., countable) $\forall\ \varepsilon \in \mathcal{E}$.

Definition 1.11. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $\widetilde{S}TS$ and consider a family of soft sets covering $\widetilde{\mathcal{X}}$, denoted by $\widetilde{O} = \{(O_\lambda\ , \mathcal{E}) : \lambda \in \Lambda\}$. The space $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is recognized as:

1. Soft compact ($soft_{Com.}$) [10] [7], soft semi-compact ($soft_{semi-Com.}$) [16], soft pre-compact ($soft_{pre-Com.}$) [18], or soft α-compact ($soft_{\alpha-Com.}$) [19] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{\mathcal{T}}$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} S O(\widetilde{\mathcal{X}})$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} PO(\widetilde{\mathcal{X}})$, or $\widetilde{O} \widetilde{\subseteq} \widetilde{S} \alpha O(\widetilde{\mathcal{X}})$, respectively, possesses a finite soft subcover.
2. Soft Lindelöf ($soft_{Lind.}$) [10] [7], soft semi-Lindelöf ($soft_{semi-Lind.}$) [17], soft pre-Lindelöf ($soft_{pre-Lind.}$) [18], or soft α-Lindelöf ($soft_{\alpha-Lind.}$) [19] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{\mathcal{T}}$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} S O(\widetilde{\mathcal{X}})$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} PO(\widetilde{\mathcal{X}})$, or $\widetilde{O} \widetilde{\subseteq} \widetilde{S} \alpha O(\widetilde{\mathcal{X}})$, respectively, possesses a countable soft subcover.
3. Soft_{int} compact ($\text{soft}_{\text{int Com.}}$) or soft_{int} Lindelöf ($\text{soft}_{\text{int Lind.}}$) [22] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{S} RC(\widetilde{\mathcal{X}})$ possesses a finite or countable soft subcollection with $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n} \widetilde{s}int(O_{\lambda_i}, \mathcal{E})$ or $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty} \widetilde{s}int(O_{\lambda_i}, \mathcal{E})$, respectively.
4. Soft nearly compact ($soft_{N.\ Com.}$) or soft nearly Lindelöf ($soft_{N.\ Lind.}$) [21] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{S} RO(\widetilde{\mathcal{X}})$ possesses a finite or countable soft subcover, respectively.
5. Mildly soft compact [20], mildly semi-compact [17], or mildly α-compact [19] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{S} CO(\widetilde{\mathcal{X}})$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} S CO(\widetilde{\mathcal{X}})$, or $\widetilde{O} \widetilde{\subseteq} \widetilde{S} \alpha CO(\widetilde{\mathcal{X}})$, respectively, possesses a finite soft subcover.
6. Mildly soft Lindelöf [20], mildly semi-Lindelöf [17], or mildly α-Lindelöf [19] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{S} CO(\widetilde{\mathcal{X}})$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} S CO(\widetilde{\mathcal{X}})$, or $\widetilde{O} \widetilde{\subseteq} \widetilde{S} \alpha CO(\widetilde{\mathcal{X}})$, respectively, possesses a countable soft subcover.
7. Almost soft compact [20], almost semi-compact [17], or almost α-compact [19] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{\mathcal{T}}$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} S O(\widetilde{\mathcal{X}})$, or $\widetilde{O} \widetilde{\subseteq} \widetilde{S} \alpha O(\widetilde{\mathcal{X}})$, respectively, possesses a finite soft subcollection with $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n} \widetilde{s}cl(O_{\lambda_i}, \mathcal{E})$, $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n} \widetilde{s}scl(O_{\lambda_i}, \mathcal{E})$, or $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n} \widetilde{s}\alpha cl(O_{\lambda_i}, \mathcal{E})$, respectively.
8. Almost soft Lindelöf [20], almost semi-Lindelöf [17], or almost α-Lindelöf [19] if every soft cover $\widetilde{O} \widetilde{\subseteq} \widetilde{\mathcal{T}}$, $\widetilde{O} \widetilde{\subseteq} \widetilde{S} S O(\widetilde{\mathcal{X}})$, or $\widetilde{O} \widetilde{\subseteq} \widetilde{S} \alpha O(\widetilde{\mathcal{X}})$, respectively, possesses a countable soft subcollection with $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty} \widetilde{s}cl(O_{\lambda_i}, \mathcal{E})$, $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty} \widetilde{s}scl(O_{\lambda_i}, \mathcal{E})$, or $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty} \widetilde{s}\alpha cl(O_{\lambda_i}, \mathcal{E})$, respectively.

Definition 1.12. Consider a family of soft sets denoted by $\widetilde{O} = \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$. This family is recognized to possess:

1. The finite soft intersection property (abbreviated, FSIP) if the soft intersection of any arbitrary finite subfamily drawn from $\widetilde{O}$ yields a non-null soft set [7].
2. The countable soft intersection property (abbreviated, CSIP) if the soft intersection of any arbitrary countable subfamily drawn from $\widetilde{O}$ yields a non-null soft set [17].

Proposition 1.13. ([16], [18], [19], [17], [20]) In every $\widetilde{S}TS$, each of the following implications holds:

1. Every $soft_{Com.}$ or $soft_{Lind.}$ space is mildly soft compact or mildly soft Lindelöf, respectively.
2. Every $soft_{semi-Com.}$ or $soft_{semi-Lind.}$ space is mildly soft semi-compact or mildly soft semi-Lindelöf, respectively.
3. Every $soft_{\alpha-Com.}$ or $soft_{\alpha-Lind.}$ space is mildly soft α-compact or mildly soft α-Lindelöf, respectively.

Proposition 1.14. ([16], [17], [18], [19], [20]) In every $\widetilde{S}TS$, each of the following implications holds:

1. Every $soft_{Com.}$ or $soft_{Lind.}$ space is almost soft compact or almost soft Lindelöf, respectively.
2. Every $soft_{semi-Com.}$ or $soft_{semi-Lind.}$ space is almost soft semi-compact or almost soft semi-Lindelöf, respectively.
3. Every $soft_{\alpha-Com.}$ or $soft_{\alpha-Lind.}$ space is almost soft α-compact or almost soft α-Lindelöf, respectively.

2. Soft S_p-compact and soft S_p-Lindelöf spaces

In this section, we introduce soft S_p-compact and soft S_p-Lindelöf spaces via the class of soft S_p-open covers of a $\widetilde{S}TS$, develop their fundamental properties, and locate them within the hierarchy of compactness-type notions surveyed in Section 1. Illustrative examples are included to show that the resulting notions are proper generalizations.

Definition 2.1. A family $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ of soft S_p-open sets in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is called a soft S_p-open cover of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$, if $\widetilde{\mathcal{X}} = \widetilde{\cup}_{\lambda \in \Lambda}(O_\lambda, \mathcal{E})$, and written $\widetilde{S} S_p O(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$-cover or $\widetilde{S} S_p O(\widetilde{\mathcal{X}})$-cover.

Definition 2.2. A $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is called a soft S_p-compact ($soft_{S_p-Com.}$) (resp., soft S_p-Lindelöf ($soft_{S_p-Lind.}$)) space, if every $\widetilde{S} S_p O(\widetilde{\mathcal{X}})$-cover $\widetilde{O} = \{(O_\lambda\ , \mathcal{E}) : \lambda \in \Lambda\}$ of $\widetilde{\mathcal{X}}$, there is a finite (resp., countable) soft subcollection of $\widetilde{O}$ such that $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n}(O_{\lambda_i}, \mathcal{E})$ (resp., $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$).

Proposition 2.3. For every $\widetilde{S}TS$ the following are true:

1. Every $soft_{semi-Com.}$ (resp., $soft_{semi-Lind.}$) space is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

2. Every $soft_{S_p-Com.}$ space is $soft_{S_p-Lind.}$.

Proof. (1) We establish the result for $soft_{S_p-Lind.}$, the argument for $soft_{S_p-Com.}$ follows mutatis mutandis.

Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $soft_{semi-Lind.}$ space and $\widetilde{O} = \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an arbitrary $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. Then, $\widetilde{O}$ is an $\widetilde{S}SO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. Since $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{semi-Lind}$, there exists a countable soft subcollection $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty} \subseteq \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$. Thus, $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}$ forms a countable soft subcover of $\widetilde{\mathcal{X}}$. Therefore, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Lind}$.

(2) Immediate from Definition 2.2.

The following examples illustrate that the converses of both parts of Proposition 2.3 fail in general. Intuitively, this stems from the broader structural nature of soft S_p-open sets compared to soft semi-open sets (as shown in Example 2.4), and the fundamental difference between requiring finite versus countable covering reductions (as shown in Example 2.5).

Example 2.4. Let $\mathcal{X} = \mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\mathcal{E} = \{\varepsilon\}$ and let $\widetilde{\varepsilon}_0 = (\varepsilon, \{0\})$ be a fixed soft point. Define the soft topology $\widetilde{\mathcal{T}}$ on $\widetilde{\mathcal{X}}$ by $\widetilde{\mathcal{T}} = \left\{\widetilde{\varnothing}\right\} \cup \{\widetilde{\mho} \widetilde{\subseteq} \widetilde{\mathcal{X}} : \widetilde{\varepsilon}_0 \widetilde{\in} \widetilde{\mho}\}$.

- Soft S_p-Compactness: Let $\widetilde{O} = \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be any $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. Then, there is $\widetilde{\varepsilon}_0 \widetilde{\in} (O_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E}) \widetilde{\in} \widetilde{O}$. By Definition 1.2 there is $(B_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E}) \widetilde{\in} \widetilde{S}PC(\widetilde{\mathcal{X}})$ with $\widetilde{\varepsilon}_0 \widetilde{\in} (B_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E}) \widetilde{\subseteq} (O_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E})$. Since $\widetilde{\varepsilon}_0 \widetilde{\in} \widetilde{\mathcal{T}}$, $\widetilde{\varepsilon}_0 = \widetilde{sint}(\widetilde{\varepsilon}_0) \widetilde{\subseteq} \widetilde{sint}(B_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E})$. Taking the soft closure yields $\widetilde{\mathcal{X}} = \widetilde{scl}(\widetilde{\varepsilon}_0) \widetilde{\subseteq} \widetilde{scl}(\widetilde{sint}(B_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E})) \widetilde{\subseteq} (B_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E})$, which forces $(B_{\lambda(\widetilde{\varepsilon}_0)}, \mathcal{E}) = \widetilde{\mathcal{X}}$; hence, this soft set forms a finite soft subcover. Thus, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$.
- Non-Soft Compactness and Non-Soft Semi-Compactness: Consider the collection $\widetilde{\mho} = \left\{\widetilde{\mho}_n = \{(\varepsilon, \{0, n\})\}, n \in \mathbb{N}\right\} \widetilde{\subseteq} \widetilde{\mathcal{T}} \widetilde{\subseteq} \widetilde{S}SO(\widetilde{\mathcal{X}})$ and is a cover of $\widetilde{\mathcal{X}}$. However, any finite subcollection of $\widetilde{\mho}$ covers only finitely many elements of $\mathcal{X} = \mathbb{N}_0$, leaving infinitely many points uncovered. Thus $\widetilde{\mho}$ admits no finite soft subcover, proving that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is neither $soft_{Com.}$ nor $soft_{semi-Com.}$.

Example 2.5. Let $\mathcal{X} = \mathbb{Z}$, where $\mathbb{Z}$ is the set of all integers, $\mathcal{E} = \{\varepsilon_1, \varepsilon_2\}$ and $\widetilde{\mathcal{T}}$ be the discrete soft topology on $\widetilde{\mathbb{Z}}$. Then, $(\widetilde{\mathbb{Z}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Lind.}$, but not $soft_{S_p-Com.}$.

Remark 2.6. It is worth noting that the notions of $soft_{Com.}$ and $soft_{S_p-Com.}$ are completely independent of each other. That is, a $soft_{Com.}$ space need not be $soft_{S_p-Com.}$ (as demonstrated in Example 2.7), and conversely, a $soft_{S_p-Com.}$ space need not be $soft_{Com.}$ (as demonstrated in Example 2.4). Consequently, neither property implies the other in general soft topological spaces.

Example 2.7. Let $\mathcal{X} = \mathbb{N}$, $\mathcal{E} = \{\varepsilon\}$ and $\widetilde{\mathcal{T}}_{cof}$ be the soft cofinite topology on $\widetilde{\mathcal{X}}$.

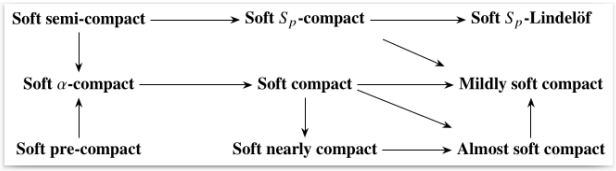


Figure 1: Relationships between soft S_p-compactness and other types of soft compact spaces.

- Soft compactness: Since the complement of every non-empty soft open is finite, every soft open cover of $\widetilde{\mathcal{X}}$ contains a finite soft subcover, proving that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}_{cof}, \mathcal{E})$ is $soft_{Com.}$.
- Non-Soft S_p-Compactness: For each soft singleton $\{(\varepsilon, \{n\})\}$, we have $\widetilde{sint}(\{(\varepsilon, \{n\})\}) = \widetilde{\varnothing}$, which implies $\widetilde{scl}(\widetilde{sint}(\{(\varepsilon, \{n\})\})) = \widetilde{\varnothing} \widetilde{\subseteq} \{(\varepsilon, \{n\})\}$. Thus, every soft singleton is soft pre-closed. Consequently, for any $(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} \widetilde{\mathcal{X}}$ and any $\widetilde{\varepsilon}_n \widetilde{\in} (\mathcal{M}, \mathcal{E})$, taking $(\mathcal{C}, \mathcal{E}) = \widetilde{\varepsilon}_n \widetilde{\in} \widetilde{S}PC(\widetilde{\mathcal{X}})$ such that $\widetilde{\varepsilon}_n \widetilde{\in} (\mathcal{C}, \mathcal{E}) \widetilde{\subseteq} (\mathcal{M}, \mathcal{E})$, which forces $\widetilde{S}S_pO\left(\widetilde{\mathcal{X}}\right) = \mathcal{P}(\widetilde{\mathcal{X}})$. The collection $\widetilde{\mho} = \{\{(\varepsilon, \{n\})\}, n \in \mathbb{N}\}$ is a $\widetilde{S}S_pO\left(\widetilde{\mathcal{X}}\right)$-cover of $\widetilde{\mathcal{X}}$. Since $\mathcal{X} = \mathbb{N}$ is infinite and the elements of $\widetilde{\mho}$ are pairwise disjoint singletons, $\widetilde{\mho}$ admits no finite soft subcover. Therefore, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}_{cof}, \mathcal{E})$ is $soft_{Com.}$, but not $soft_{S_p-Com.}$.

Proposition 2.8. If $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com}$ (resp., $soft_{S_p-Lind}$) space, then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is mildly soft compact (resp., mildly soft Lindelöf).

Proof. We establish the result for $soft_{S_p-Lind.}$ spaces; the argument for $soft_{S_p-Com.}$ spaces follows mutatis mutandis.

Let $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an arbitrary $\widetilde{S}CO(\widetilde{\mathcal{X}})$-cover. By Proposition 3.13(4) of Ref. [25], $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. Since $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Lind.}$, there exists a countable soft subcollection $\{(O_{\lambda_i'}, \mathcal{E})\}_{i=1}^{\infty} \subseteq \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$. Thus, $\left\{(O_{\lambda_i}, \mathcal{E})\right\}_{i=1}^{\infty}$ forms a countable soft subcover of $\widetilde{\mathcal{X}}$. Therefore, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is mildly soft Lindelöf.

The interrelationships between soft S_p-compactness and various other forms of soft compactness (established via Proposition 2.3, Proposition 2.8, and existing literature in Refs. [19–21]) are illustrated in Figure 1.

Theorem 2.9. A $\widetilde{STS}$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) if and only if every family of soft S_p-closed subsets of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ satisfying FSIP (resp., CSIP) has a non-null soft intersection.

Proof. We restrict the proof to the case where $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Lind.}$, as the other case proceeds mutatis mutandis.

Let $\mathcal{F} = \{(C_\lambda, \mathcal{E}) : \lambda \in \Lambda\} \widetilde{\subseteq} \widetilde{S}S_pC(\widetilde{\mathcal{X}})$ satisfy the CSIP. Suppose, for a contradiction, that $\widetilde{\cap}_{\lambda \in \Lambda}(C_\lambda, \mathcal{E}) = \widetilde{\varnothing}$. Then, $\widetilde{\cup}_{\lambda \in \Lambda}(\widetilde{\mathcal{X}} \widetilde{\setminus} (C_\lambda, \mathcal{E})) = \widetilde{\mathcal{X}}$. The collection $\{\widetilde{\mathcal{X}} \widetilde{\setminus} (C_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ forms a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. By soft S_p-Lindelöfness, there is a countable soft subcollection $\{\widetilde{\mathcal{X}} \widetilde{\setminus} (C_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty} \subseteq$

$\{\widetilde{\mathcal{X}}\backslash(C_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty}\widetilde{\mathcal{X}}\backslash(C_{\lambda_i}, \mathcal{E})$. Therefore, $\widetilde{\varnothing} = \widetilde{\cap}_{i=1}^{\infty}(C_{\lambda_i}, \mathcal{E})$. This contradicts the hypothesis that $\mathcal{F}$ satisfies the CSIP. Thus, $\widetilde{\cap}_{\lambda\in\Lambda}(C_\lambda, \mathcal{E}) \neq \widetilde{\varnothing}$.

Conversely, let $\widetilde{\mathcal{O}} = \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an arbitrary $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. Then, $\widetilde{\mathcal{X}} = \widetilde{\cup}_{\lambda\in\Lambda}(\mathcal{O}_\lambda, \mathcal{E})$, which implies that $\widetilde{\cap}_{\lambda\in\Lambda}\widetilde{\mathcal{X}}\backslash(\mathcal{O}_\lambda, \mathcal{E}) = \widetilde{\varnothing}$. Thus, $\{\widetilde{\mathcal{X}}\backslash(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\subseteq}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$ cannot satisfy the CSIP. By the hypothesis, there must exist a countable soft subcollection $\{\widetilde{\mathcal{X}}\backslash(\mathcal{O}_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}$ such that $\widetilde{\cap}_{i=1}^{\infty}\widetilde{\mathcal{X}}\backslash(\mathcal{O}_{\lambda_i}, \mathcal{E}) = \widetilde{\varnothing}$. Taking complements, we get $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Hence, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Lind.}$.

Proposition 2.10. If $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space, then every $\widetilde{S}\theta O(\widetilde{\mathcal{X}})$-cover, $\widetilde{S}RO(\widetilde{\mathcal{X}})$-cover or $\widetilde{S}CO(\widetilde{\mathcal{X}})$-cover has a finite (resp., countable) soft subcover.

Proof. By Proposition 3.13 of Ref. [25], every $\widetilde{S}\theta O(\widetilde{\mathcal{X}})$, $\widetilde{S}RO(\widetilde{\mathcal{X}})$ or $\widetilde{S}CO(\widetilde{\mathcal{X}})$ is $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$. Hence, the given cover is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$, and the conclusion follows from Definition 2.2.

Proposition 2.11. If every $\widetilde{S}PC(\widetilde{\mathcal{X}})$-cover has a finite (resp., countable) soft subcover, then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space.

Proof. Let $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an arbitrary $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. Then, $\forall\ \widetilde{\varepsilon_x}\widetilde{\in}\widetilde{S}P(\widetilde{\mathcal{X}})$, pick $\lambda(\widetilde{\varepsilon_x}) \in \Lambda$ with $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E})$, by Definition 1.2 there is $(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E})\widetilde{\in}\widetilde{S}PC(\widetilde{\mathcal{X}})$ with $\widetilde{\varepsilon_x}\widetilde{\in}(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E})\widetilde{\subseteq}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E})$. The family, $\{(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E}) : \widetilde{\varepsilon_x}\widetilde{\in}\widetilde{S}P(\widetilde{\mathcal{X}})\}$ is a $\widetilde{S}PC(\widetilde{\mathcal{X}})$-cover. By hypothesis, this family has a finite (resp., countable) soft subcover, that is $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{n}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})$ (resp., $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{\infty}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})$). Therefore, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proposition 2.12. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be $soft_{Lo.\ Ind.}$ or $soft_{T_1-space}$. Then, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space if and only if $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{semi-Com.}$ (resp., $soft_{semi-Lind.}$).

Proof. Suppose that $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$, and let $\widetilde{\mathcal{O}} = \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}SO(\widetilde{\mathcal{X}})$-cover. As $\widetilde{\mathcal{X}}$ is $soft_{Loc\ Ind}$ or $soft_{T_1-space}$, Theorem 3.17 and Theorem 3.19 of Ref. [25] yields $\widetilde{\mathcal{O}}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. The soft S_p-compactness of $\widetilde{\mathcal{X}}$ therefore yields a finite subcover, whence $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{semi-Com}$.

Conversely, this is Proposition 2.3(1).

The Lindelöf case is analogous, mutatis mutandis.

Proposition 2.13. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $soft_{Lo.\ Ind.}$ space. Then, the following statements are equivalent for each of the compact and Lindelöf versions:

1. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space
2. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{Com.}$ (resp., $soft_{Lind.}$).
3. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{\alpha-Com.}$ (resp., $soft_{\alpha-Lind.}$).

Proof. By Corollary 3.18(1-2) of Ref. [25], $\widetilde{\mathcal{T}} = \widetilde{S}\alpha O(\widetilde{\mathcal{X}}) = \widetilde{S}S_pO(\widetilde{\mathcal{X}})$, so a family of soft sets is soft open, soft α-open, or soft S_p-open simultaneously. The equivalences (1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) follow at once.

Proposition 2.14. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $soft_{Lo.\ Ind.}$ space. Then, every soft pre-compactness (respectively, soft pre-Lindelöfness) implies soft S_p-compactness (respectively, soft S_p-Lindelöfness).

Proof. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be $soft_{pre-Com.}$ (resp., $soft_{pre-Lind.}$). By Corollary 3.18(3) of Ref. [25], every $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover is a $\widetilde{S}PO(\widetilde{\mathcal{X}})$-cover, and admits a finite (respectively, countable) soft subcover.

Proposition 2.15. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $soft_{T_1-space}$ and a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space. Then, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is both $soft_{Com.}$ and $soft_{\alpha-Com.}$ (resp., $soft_{Lind.}$, and $soft_{\alpha-Lind.}$).

Proof. Let $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be a soft open (or $\widetilde{S}\alpha O(\widetilde{\mathcal{X}})$-) cover of $\widetilde{\mathcal{X}}$. As $\widetilde{\mathcal{X}}$ is a $soft_{T_1-space}$, Corollary 3.20(1-2) of Ref. [25] yields that $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. By the soft S_p-compactness of $\widetilde{\mathcal{X}}$, there is $\{(\mathcal{O}_{\lambda_i}, \mathcal{E})\}_{i=1}^{n}$ with $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Thus, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is both $soft_{Com.}$ and $soft_{\alpha-Com.}$.

The Lindelöf case is analogous, mutatis mutandis.

Proposition 2.16. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $soft_{E.\ D.}$ space. Then,

1. if $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$), then it is $soft_{N.\ Com.}$ (resp., $soft_{N.\ Lind.}$).
2. if $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{pre-Com.}$ (resp., $soft_{pre-Lind.}$), then it is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).
3. if $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{\alpha-Com.}$ (resp., $soft_{\alpha-Lind.}$), then it is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proof. (1) By Proposition 3.25 in [25], every $\widetilde{S}RO(\widetilde{\mathcal{X}})$-cover is $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover, apply Definition 2.2.

(2) By Theorem 3.22 of Ref. [25], every $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover is $\widetilde{S}PO(\widetilde{\mathcal{X}})$-cover, apply Definition of $soft_{pre-Com.}$ (resp., $soft_{pre-Lind.}$).

(3) By Proposition 3.23 of Ref. [25], every $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover is $\widetilde{S}\alpha O(\widetilde{\mathcal{X}})$-cover, apply Definition of $soft_{\alpha-Com.}$ (resp., $soft_{\alpha-Lind.}$).

Proposition 2.17. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be $soft_{Lo.\ Ind.}$ (or $soft_{T_1-space}$), and $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$). Then,

1. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is mildly soft α-compact (resp., mildly soft α-Lindelöf).
2. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is mildly soft semi-compact (resp., mildly soft semi-Lindelöf).

Proof. (1) Combine Proposition 2.13 (or Proposition 2.15) with Proposition 1.13(3).

(2) Combine Proposition 2.12 with Proposition 1.13(2).

Proposition 2.18. Under the same hypotheses as Proposition 2.17, then:

1. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is almost soft compact (resp., almost soft Lindelöf).

2. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is almost soft α-compact (resp., almost soft α-Lindelöf).
3. $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is almost soft semi-compact (resp., almost soft semi-Lindelöf).

Proof. (1) Apply Proposition 2.13 (or Proposition 2.15) together with Proposition 1.14(1).

(2) Apply Proposition 2.13 (or Proposition 2.15) together with Proposition 1.14(3).

(3) Apply Proposition 2.12 together with Proposition 1.14(2).

Proposition 2.19. If $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a soft p-Alexandroff space and $\text{soft}_{\text{int Com.}}$ (resp., $\text{soft}_{\text{int Lind.}}$), then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proof. Let $\widetilde{\mathcal{O}} = \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. By Proposition 3.31 of Ref. [25], $\widetilde{S}RC(\widetilde{\mathcal{X}}) = \widetilde{S}S_pO(\widetilde{\mathcal{X}})$, so $\widetilde{\mathcal{O}}$ is a $\widetilde{S}RC(\widetilde{\mathcal{X}})$-cover. By the soft_{int} compactness of $\widetilde{\mathcal{X}}$, there is a finite soft subcollection with $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n}\widetilde{int}(\mathcal{O}_{\lambda_i}, \mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{n}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Therefore, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$.

The Lindelöf case is analogous.

Lemma 2.20. If $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{E.\ D.}$ space, then

1. $(\mathcal{O}, \mathcal{E}) = \widetilde{int}(\mathcal{O}, \mathcal{E}), \forall\ (\mathcal{O}, \mathcal{E})\widetilde{\in}\widetilde{S}RC(\widetilde{\mathcal{X}})$.
2. $(E, \mathcal{E}) = \widetilde{scl}(E, \mathcal{E}), \forall\ (E, \mathcal{E})\widetilde{\in}\widetilde{S}RO(\widetilde{\mathcal{X}})$.

Proof. (1) Let $(\mathcal{O}, \mathcal{E})\widetilde{\in}\widetilde{S}RC(\widetilde{\mathcal{X}})$. Then, $(\mathcal{O}, \mathcal{E}) = \widetilde{scl}(\widetilde{int}(\mathcal{O}, \mathcal{E}))$. Since $\widetilde{int}(\mathcal{O}, \mathcal{E})$ is a soft closed set under $soft_{E.\ D.}$, so $\widetilde{int}(\mathcal{O}, \mathcal{E}) = \widetilde{scl}(\widetilde{int}(\mathcal{O}, \mathcal{E})) = (\mathcal{O}, \mathcal{E})$.

(2) The proof follows the same pattern as (1).

Proposition 2.21. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be $soft_{E.\ D.}$ and $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$). Then, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $\text{soft}_{\text{int Com.}}$ (resp., $\text{soft}_{\text{int Lind.}}$).

Proof. Let $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be a $\widetilde{S}RC(\widetilde{\mathcal{X}})$-cover. By Proposition 3.13(3) of Ref. [25], this is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover, and the soft S_p-compactness of $\widetilde{\mathcal{X}}$ yields a finite soft subcollection with $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Applying Lemma 2.20(1) to each $(\mathcal{O}_\lambda, \mathcal{E})$, we have $\widetilde{\mathcal{X}} = \widetilde{\cup}_{i=1}^{n}\widetilde{int}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Thus, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $\text{soft}_{\text{int Com}}$.

The Lindelöf case is analogous.

3. Soft S_p-(compactness and Lindelöfness) sets and subspaces

In this section, we develop the local theory of soft S_p-compactness and soft S_p-Lindelöfness, introducing soft S_p-compact and soft S_p-Lindelöf sets, together with their subspace counterparts, and studying their behaviour under soft clopen and soft S_p-closed subsets and under finite soft unions and intersections.

Definition 3.1. $(\mathcal{M}, \mathcal{E})\widetilde{\subseteq}(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is called:

1. $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set, if every $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover $\widetilde{\mathcal{O}} = \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ of $(\mathcal{M}, \mathcal{E})$, there is a finite (resp., countable) soft subcollection of $\widetilde{\mathcal{O}}$ with $(\mathcal{M}, \mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{n}(\mathcal{O}_{\lambda_i}, \mathcal{E})$ (resp., $(\mathcal{M}, \mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E})$).
2. Relative $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set, if the soft subspace $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Theorem 3.2. Let $(\mathcal{M}, \mathcal{E})$ and $(B, \mathcal{E})$ be two soft subsets of a $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ such that $(B, \mathcal{E})\widetilde{\subseteq}(\mathcal{M}, \mathcal{E})$ and $(\mathcal{M}, \mathcal{E})\widetilde{\in}\widetilde{S}CO(\widetilde{\mathcal{X}})$. Then, $(B, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ if and only if it is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$.

Proof. Suppose that $(B, \mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$, and let $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E}))$-cover of $(B, \mathcal{E})$. $\forall\lambda \in \Lambda$ there is $(\mathcal{R}_\lambda, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$ such that $(\mathcal{O}_\lambda, \mathcal{E}) = (\mathcal{R}_\lambda, \mathcal{E})\widetilde{\cap}(\mathcal{M}, \mathcal{E})$. Thus, $\{(\mathcal{R}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ is an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(B, \mathcal{E})$. Since $(B, \mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $\widetilde{\mathcal{X}}$, there is a countable soft subcollection $\left\{(\mathcal{R}_{\lambda_i}, \mathcal{E})\right\}_{i=1}^{\infty} \subseteq \{(\mathcal{R}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $(B, \mathcal{E})\widetilde{\subseteq}\ \widetilde{\cup}_{i=1}^{\infty}(\mathcal{R}_{\lambda_i}, \mathcal{E})$. Thus,

$$\begin{aligned}(B, \mathcal{E})\widetilde{\subseteq}&\left(\widetilde{\cup}_{i=1}^{\infty}(\mathcal{R}_{\lambda_i}, \mathcal{E})\right)\widetilde{\cap}(\mathcal{M}, \mathcal{E})\\ &= \widetilde{\cup}_{i=1}^{\infty}\left((\mathcal{R}_{\lambda_i}, \mathcal{E})\widetilde{\cap}(\mathcal{M}, \mathcal{E})\right)\\ &= \widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E}).\end{aligned}$$

Hence, $(B, \mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Thus, $\{(\mathcal{O}_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}$ is a countable soft subcover of $(B, \mathcal{E})$ in $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$. Consequently, $(B, \mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$.

Conversely, let $(B, \mathcal{E})$ be a $soft_{S_p-Lind.}$ set in $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$ and let $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(B, \mathcal{E})$. As $(\mathcal{M}, \mathcal{E})\widetilde{\in}\widetilde{S}CO(\widetilde{\mathcal{X}})$, Proposition 3.4 of Ref. [27] implies that $(\mathcal{M}, \mathcal{E})\widetilde{\cap}(\mathcal{O}_\lambda, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E}))$, $\forall\ \lambda \in \Lambda$. Thus, $\left\{(\mathcal{M}, \mathcal{E})\widetilde{\cap}(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\right\}$ is a $\widetilde{S}S_pO(((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E}))$-cover of $(B, \mathcal{E})$. By hypothesis, there is a countable soft subcollection $\{(\mathcal{O}_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty} \subseteq \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $(B, \mathcal{E})\widetilde{\subseteq}\ \widetilde{\cup}_{i=1}^{\infty}(\mathcal{M}, \mathcal{E})\widetilde{\cap}(\mathcal{O}_{\lambda_i}, \mathcal{E})\widetilde{\subseteq}\ \widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E})$. Therefore, $(B, \mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

The $soft_{S_p-Com.}$ case follows by an analogous argument, mutatis mutandis.

Remark 3.3. This equivalence provides a practical tool for analyzing relative soft topologies, as it allows one to verify the soft S_p-compactness (resp., soft S_p-Lindelöfness) of a subspace directly within the ambient $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ without needing to explicitly reconstruct and evaluate soft S_p-open covers in the induced relative topology.

Corollary 3.4. A soft clopen subset $(\mathcal{M}, \mathcal{E})$ of $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set if and only if $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proposition 3.5. If $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space and $(\mathcal{M}, \mathcal{E})\widetilde{\in}\widetilde{S}CO(\widetilde{\mathcal{X}})$, then $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proof. We establish the result for $soft_{S_p-Lind}$ spaces; the argument for $soft_{S_p-Com.}$ spaces follows mutatis mutandis.

Let $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E}))$-cover of $(\mathcal{M}, \mathcal{E})$. By Corollary 4.27 of Ref. [25], $\forall\lambda \in \Lambda\ (O_\lambda, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$. Since $(\mathcal{M}, \mathcal{E})\widetilde{\in}\widetilde{S}CO(\widetilde{\mathcal{X}})$, $\widetilde{\mathcal{X}} \setminus (\mathcal{M}, \mathcal{E})\widetilde{\in}\widetilde{S}CO(\widetilde{\mathcal{X}})$. By Proposition 3.13(4) of Ref. [25], $\widetilde{\mathcal{X}} \setminus (\mathcal{M}, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$, which implies that $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{\widetilde{\mathcal{X}} \setminus (\mathcal{M}, \mathcal{E})\}$ forms a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. Since $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Lind.}$, there exists a countable soft subcollection $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty} \subseteq \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $\widetilde{\mathcal{X}} = (\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(\widetilde{\mathcal{X}} \setminus (\mathcal{M}, \mathcal{E}))$. So,

$$(\mathcal{M}, \mathcal{E}) = (\mathcal{M}, \mathcal{E})\widetilde{\cap}\widetilde{\mathcal{X}} = (\mathcal{M}, \mathcal{E})\widetilde{\cap}\left((\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(\widetilde{\mathcal{X}} \setminus (\mathcal{M}, \mathcal{E}))\right)$$

$$= [(\mathcal{M}, \mathcal{E})\widetilde{\cap}(\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))]\widetilde{\cup}[(\mathcal{M}, \mathcal{E})\widetilde{\cap}(\widetilde{\mathcal{X}} \setminus (\mathcal{M}, \mathcal{E}))]$$

$$= (\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}\widetilde{\varnothing} = \widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}).$$

Thus, $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}$ is a countable soft subcover of $(\mathcal{M}, \mathcal{E})$. Therefore, $((\mathcal{M}, \mathcal{E}), \widetilde{\mathcal{T}}_{(\mathcal{M}, \mathcal{E})}, \mathcal{E})$ is $soft_{S_p-Lind.}$.

Proposition 3.6. A $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) if and only if every proper soft S_p-closed subset of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Proof. We establish the result for $soft_{S_p-Lind.}$ spaces; the argument for $soft_{S_p-Com.}$ spaces follows mutatis mutandis.

Let $(\mathcal{C},\mathcal{E})$ be a proper soft S_p-closed subset of $soft_{S_p-Lind.}$ space $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ and let $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{C},\mathcal{E})$. Since $(\mathcal{C},\mathcal{E})\widetilde{\in}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$, $\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$. Thus, the collection $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E})\}$ forms a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$. The soft S_p-Lindelöfness of $\widetilde{\mathcal{X}}$ yields a countable soft subcollection $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}\widetilde{\cup}\{\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E})\} \subseteq \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E})\}$ such that $\widetilde{\mathcal{X}} = (\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E}))$. This implies that $(\mathcal{C},\mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$. Hence, $(\mathcal{C},\mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Conversely, suppose that every proper soft S_p-closed subset of $\widetilde{\mathcal{X}}$ is a $soft_{S_p-Lind.}$ set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$. Let $\widetilde{\mathcal{O}} = \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}}$, so $\widetilde{\mathcal{X}} = \widetilde{\cup}\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$. If $(O_\lambda, \mathcal{E}) = \widetilde{\mathcal{X}}$ for some λ, then $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ has a countable soft subcover of $\widetilde{\mathcal{X}}$. Otherwise, assume $\widetilde{\mathcal{X}} \neq (O_\lambda, \mathcal{E}) \neq \widetilde{\varnothing}$ $\forall\ \lambda \in \Lambda$. Fix a specific $(O_\delta, \mathcal{E})\widetilde{\in}\widetilde{\mathcal{O}}$ for some $\delta \in \Lambda$. Then, $(\mathcal{C}, \mathcal{E}) = \widetilde{\mathcal{X}} \setminus (O_\delta, \mathcal{E})\widetilde{\in}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$ and $\widetilde{\mathcal{X}} \neq (\mathcal{C},\mathcal{E}) \neq \widetilde{\varnothing}$. Since $(\mathcal{C}, \mathcal{E})\widetilde{\subseteq}\widetilde{\mathcal{X}}$, the collection $\widetilde{\mathcal{O}}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{C}, \mathcal{E})$. By hypothesis, $(\mathcal{C}, \mathcal{E})$ is a $soft_{S_p-Lind.}$ set, so there is a countable soft subcollection $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty} \subseteq \widetilde{\mathcal{O}}$ such that $(\mathcal{C}\,\mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$. Thus, $\widetilde{\mathcal{X}} = (\mathcal{C}, \mathcal{E})\widetilde{\cup}(O_\delta, \mathcal{E})\widetilde{\subseteq}(\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(O_\delta, \mathcal{E})$. This provides a countable soft subcover $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}\widetilde{\cup}(O_\delta, \mathcal{E})$ of $\widetilde{\mathcal{X}}$. Hence, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is *a* $soft_{S_p-Lind.}$ space.

Corollary 3.7. Every proper soft θ-closed, soft regular open, or soft clopen subset of $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set.

Proof. Immediate from Proposition 2.5 of Ref. [26] and Proposition 3.6.

Proposition 3.8. If $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ and $(\mathcal{C},\mathcal{E})\widetilde{\in}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$, then $(\mathcal{M},\mathcal{E})\widetilde{\cap}(\mathcal{C},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Proof. We establish the proof for $soft_{S_p-Lind.}$ sets; the argument for $soft_{S_p-Com.}$ sets follows mutatis mutandis.

Let $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})\widetilde{\cap}(\mathcal{C},\mathcal{E})$. Since $(\mathcal{C},\mathcal{E})\widetilde{\in}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$, $\widetilde{\mathcal{X}}\setminus(\mathcal{C}, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$. Observe that: $(\mathcal{M},\mathcal{E})\widetilde{\subseteq}(\widetilde{\cup}_{\lambda\in\Lambda}(O_\lambda, \mathcal{E}))\widetilde{\cup}(\widetilde{\mathcal{X}}\setminus(\mathcal{C}, \mathcal{E}))$, so the family $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{\widetilde{\mathcal{X}}\setminus(\mathcal{C}, \mathcal{E})\}$ forms a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})$. As $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Lind}$ set, there is a countable soft subcollection $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}\widetilde{\cup}\{\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E})\} \subseteq \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{\widetilde{\mathcal{X}} \setminus (\mathcal{C}, \mathcal{E})\}$ such that $(\mathcal{M},\mathcal{E})\widetilde{\subseteq}(\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(\widetilde{\mathcal{X}}\setminus(\mathcal{C}, \mathcal{E}))$. Thus, $(\mathcal{M},\mathcal{E})\widetilde{\cap}(\mathcal{C},\mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$. So, $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}$ is a countable soft subcover of $(\mathcal{M},\mathcal{E})\widetilde{\cap}(\mathcal{C},\mathcal{E})$. Hence, $(\mathcal{M},\mathcal{E})\widetilde{\cap}(\mathcal{C},\mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Corollary 3.9. If $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ and $(\mathcal{C},\mathcal{E})$ is a soft θ-closed, soft regular open, or soft clopen subset of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$, then $(\mathcal{M},\mathcal{E})\widetilde{\cap}(\mathcal{C},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Proof. Combine Proposition 2.5 of Ref. [26] with Proposition 3.8.

Proposition 3.10. If $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ and $(B,\mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$ with $(B,\mathcal{E})\widetilde{\subseteq}(\mathcal{M},\mathcal{E})$, then $(\mathcal{M},\mathcal{E})\widetilde{\setminus}(B,\mathcal{E})$ a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Proof. We establish the proof for $soft_{S_p-Lind.}$ sets; the argument for $soft_{S_p-Com.}$ sets follows mutatis mutandis.

Let $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})\widetilde{\setminus}(B,\mathcal{E})$. Then, the family $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{(B, \mathcal{E})\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})$. By hypothesis, there is a countable soft subcollection $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}\widetilde{\cup}\{(B, \mathcal{E})\} \subseteq \{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}\widetilde{\cup}\{(B, \mathcal{E})\}$ such that $(\mathcal{M},\mathcal{E})\widetilde{\subseteq}(\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(B, \mathcal{E})$. Thus, $(\mathcal{M},\mathcal{E})\widetilde{\setminus}(B,\mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{\infty}(O_{\lambda_i}, \mathcal{E})$. So, $\{(O_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}$ is a countable soft subcover of $(\mathcal{M},\mathcal{E})\widetilde{\setminus}(B,\mathcal{E})$. Hence, $(\mathcal{M},\mathcal{E})\widetilde{\setminus}(B,\mathcal{E})$ is a $soft_{S_p-Lind.}$ set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Corollary 3.11. If $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ and $(B,\mathcal{E})$ is a soft θ-closed, soft regular open, or soft clopen subset of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ with $(B,\mathcal{E})\widetilde{\subseteq}(\mathcal{M},\mathcal{E})$, then $(\mathcal{M},\mathcal{E})\widetilde{\setminus}(B,\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$.

Proof. Immediate from by Proposition 3.13 of Ref. [25] and Proposition 3.10.

Proposition 3.12. A finite (resp., countable) soft union of $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) subsets in $\widetilde{S}TS$ $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is again a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$

Proof. Immediate from Definition 3.1.

Proposition 3.13. If $(\mathcal{M}, \mathcal{E})$ is a proper subset of $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ such that both $(\mathcal{M}, \mathcal{E})$ and $\widetilde{\mathcal{X}}\backslash(\mathcal{M},\mathcal{E})$ are $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) sets in $\widetilde{\mathcal{X}}$, then $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space.

Proof. We establish the proof for $soft_{S_p-Lind.}$ sets; the argument for $soft_{S_p-Com.}$ sets follows mutatis mutandis.

Let $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $\widetilde{\mathcal{X}} = (\mathcal{M},\mathcal{E})\widetilde{\cup}\widetilde{\mathcal{X}\backslash}(\mathcal{M}, \mathcal{E})$. Then, $\{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of both $(\mathcal{M},\mathcal{E})$ and $\widetilde{\mathcal{X}\backslash}(\mathcal{M}, \mathcal{E})$. Since $(\mathcal{M}, \mathcal{E})$ and $\widetilde{\mathcal{X}\backslash}(\mathcal{M},\mathcal{E})$ are $soft_{S_p-Lind.}$ sets, there are countable soft subcollection $\{(\mathcal{O}_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty} \subseteq \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ and $\{(\mathcal{O}_{\mu_j}, \mathcal{E})\}_{j=1}^{\infty} \subseteq \{(\mathcal{O}_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ such that $(\mathcal{M},\mathcal{E})\widetilde{\subseteq} \widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E})$ and $\widetilde{\mathcal{X}\backslash}(\mathcal{M}, \mathcal{E})\widetilde{\subseteq} \widetilde{\cup}_{j=1}^{\infty}(\mathcal{O}_{\mu_j}, \mathcal{E})$. Consequently, we have $\widetilde{\mathcal{X}} = (\mathcal{M},\mathcal{E})\widetilde{\cup}(\widetilde{\mathcal{X}}\widetilde{\backslash}(\mathcal{M}, \mathcal{E}))\widetilde{\subseteq}(\widetilde{\cup}_{i=1}^{\infty}(\mathcal{O}_{\lambda_i}, \mathcal{E}))\widetilde{\cup}(\widetilde{\cup}_{j=1}^{\infty}(\mathcal{O}_{\mu_j}, \mathcal{E}))$. Since $\{\lambda_i\}_{i=1}^{\infty} \cup \{\mu_j\}_{j=1}^{\infty}$ is countable, the soft subcollection $\{(\mathcal{O}_{\lambda_i}, \mathcal{E})\}_{i=1}^{\infty}\widetilde{\cup}\{(\mathcal{O}_{\mu_j}, \mathcal{E})\}_{j=1}^{\infty}$ forms a countable soft subcover of $\widetilde{\mathcal{X}}$. Hence, $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ is a $soft_{S_p-Lind.}$ space.

Proposition 3.14. Let $(\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E})$ be a $soft_{S_p-T_2-space}$ in which $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$ is closed under finite (resp., countable) soft intersection. Then, every $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set $(\mathcal{M},\mathcal{E})\widetilde{\in}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$.

Proof. We establish the proof for a $soft_{S_p-Com.}$ set; the argument for a $soft_{S_p-Lind.}$ set follows analogously by replacing finite soft subcover and finite soft intersection with countable ones.

Let $\widetilde{\varepsilon_x}\widetilde{\in}\widetilde{\mathcal{X}\backslash}(\mathcal{M},\mathcal{E})$. $\forall$ $\widetilde{\varepsilon_y}\widetilde{\in}(\mathcal{M}, \mathcal{E})$, the soft S_p-T_2-space yields disjoint soft S_p-open sets $(\mathcal{U}_y, \mathcal{E})$ and $(\mathcal{V}_y, \mathcal{E})$ such that $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{U}_y, \mathcal{E})$ and $\widetilde{\varepsilon_y}\widetilde{\in}(\mathcal{V}_y, \mathcal{E})$. So, $\{(\mathcal{V}_y, \mathcal{E}) : \widetilde{\varepsilon_y}\widetilde{\in}(\mathcal{M}, \mathcal{E})\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})$. Since $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ set, there is a finite subcollection $\{(\mathcal{V}_{y_i}, \mathcal{E})\}_{i=1}^{n}$ such that $(\mathcal{M},\mathcal{E})\widetilde{\subseteq} \widetilde{\cup}_{i=1}^{n}(\mathcal{V}_{y_i}, \mathcal{E})$. Let $(\mathcal{U},\mathcal{E}) = \widetilde{\cap}_{i=1}^{n}(\mathcal{U}_{y_i}, \mathcal{E})$. As $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$ is closed under finite soft intersection, we have $(\mathcal{U},\mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$. Since $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{U}_{y_i}, \mathcal{E})$ for all $1 \leq i \leq n$, it follows that $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{U},\mathcal{E})$. Furthermore, $(\mathcal{U},\mathcal{E})\widetilde{\cap}(\mathcal{M}, \mathcal{E})\widetilde{\subseteq}(\mathcal{U},\mathcal{E})\widetilde{\cap}(\widetilde{\cup}_{i=1}^{n}(\mathcal{V}_{y_i}, \mathcal{E}))$

$= \widetilde{\cup}_{i=1}^{n}((\mathcal{U},\mathcal{E})\widetilde{\cap}(\mathcal{V}_{y_i}, \mathcal{E}))\widetilde{\subseteq} \widetilde{\cup}_{i=1}^{n}((\mathcal{U}_{y_i},\mathcal{E})\widetilde{\cap}(\mathcal{V}_{y_i}, \mathcal{E})) = \widetilde{\varnothing}$.

Hence, $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{U},\mathcal{E})\widetilde{\subseteq}\widetilde{\mathcal{X}\backslash}(\mathcal{M},\mathcal{E})$. Consequently by Theorem 3.9 of Ref. [25], $\widetilde{\mathcal{X}\backslash}(\mathcal{M},\mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$. Therefore, $(\mathcal{M},\mathcal{E})\widetilde{\in}\widetilde{S}S_pC(\widetilde{\mathcal{X}})$.

4. Preservation of soft S_p-compactness and soft S_p-Lindelöfness

In this section, we study the behaviour of soft S_p-compact and soft S_p-Lindelöf sets and spaces under several classes of soft mappings. The results below fall into two families: preservation results, which describe how the properties transfer along soft functions, and invariance results, which recover a property on the domain from a property on the codomain.

Remark 4.1. A soft S_p-continuous function does not necessarily preserve soft S_p-compactness. That is, if $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ is a soft S_p-continuous function and $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ set in $\widetilde{\mathcal{X}}$, then $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$ is not necessarily a $soft_{S_p-Com.}$ set in $\widetilde{Y}$. The following example demonstrates this point.

Example 4.2. Let $X = Y = \mathbb{N}$ (the set of natural numbers), and $\mathcal{E} = \{\varepsilon\}$. Define the soft topology $\widetilde{\mathcal{T}} = \{\widetilde{\emptyset}, \widetilde{\mathcal{X}}\}$. Then, we have $\widetilde{S}S_pO\left(\widetilde{\mathcal{X}}\right) = \widetilde{\mathcal{T}}$. Consider $(\mathcal{M},\mathcal{E}) = \widetilde{\mathcal{X}}$. Any soft S_p-cover of $(\mathcal{M},\mathcal{E})$ in $\widetilde{\mathcal{X}}$ must contain $\widetilde{\mathcal{X}}$, which yields a finite soft subcover $\{\widetilde{\mathcal{X}}\}$. Therefore, $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ set in $\widetilde{\mathcal{X}}$.

On Y, define the soft topology $\widetilde{\sigma} = \{\widetilde{\emptyset}, \widetilde{Y}, \{(\varepsilon, \{1\})\}\}$ we have $\widetilde{S}S_pO\left(\widetilde{Y}\right) = \left\{\widetilde{\emptyset}\right\} \cup \{(\varepsilon, \mathcal{B})\widetilde{\subseteq}\widetilde{Y} : 1 \in \mathcal{B}\}$. Now define $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \mathcal{E})$, induced by $u(x) = x + 1$ for all $x \in \mathbb{N}$ and $p(\varepsilon) = \varepsilon$. The soft function $\widetilde{f_{pu}}$ is soft S_p-continuous. Since $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ set in $\widetilde{\mathcal{X}}$, $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E}) = \widetilde{f_{pu}}\left(\widetilde{\mathcal{X}}\right) = \{(\varepsilon, \mathbb{N}\backslash\{1\})\}$. Let $\mathcal{W} = \{\{(\varepsilon, \{1, n\})\} : n \in \mathbb{N}\backslash\{1\}\}$ is a $\widetilde{S}S_pO(\widetilde{Y})$-cover of $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$. Since $\mathbb{N}\backslash\{1\}$ is infinite and each set in $\mathcal{W}$ contains only one element from $\mathbb{N}\backslash\{1\}$, no finite subcollection of $\mathcal{W}$ can cover $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$. Hence, $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$ is not $soft_{S_p-Com.}$ in $\widetilde{Y}$.

Proposition 4.3. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft S_p-continuous. If $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set in $\widetilde{\mathcal{X}}$, then $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$ is a $soft_{Com.}$ (resp., $soft_{Lind.}$) set in $\widetilde{Y}$.

Proof. Let $\{(\mathcal{O}_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ be a soft open cover of $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$. Then, $\forall$ $\widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{M},\mathcal{E})$, pick $\lambda(\widetilde{\varepsilon_x}) \in \Lambda$ with $\widetilde{f_{pu}}(\widetilde{\varepsilon_x})\widetilde{\in}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})}, \acute{\mathcal{E}})$. By soft S_p-continuity, there is $\widetilde{\varepsilon_x}\widetilde{\in}(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E})\widetilde{\in}\widetilde{S}S_pO(\widetilde{\mathcal{X}})$ with $\widetilde{f_{pu}}(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E})\widetilde{\subseteq}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})}, \acute{\mathcal{E}})$. So, the family $\{(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E}) : \widetilde{\varepsilon_x}\widetilde{\in}(\mathcal{M},\mathcal{E})\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})$. As $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com}$ set, there is a finite soft subcollection with $(\mathcal{M},\mathcal{E})\widetilde{\subseteq}\widetilde{\cup}_{i=1}^{n}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})$. Thus,

$$\begin{aligned}\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})\widetilde{\subseteq}\widetilde{f_{pu}}\left(\widetilde{\cup}_{i=1}^{n}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})\right)\\ = \widetilde{\cup}_{i=1}^{n}\left(\widetilde{f_{pu}}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})\right)\\ \widetilde{\subseteq}\widetilde{\cup}_{i=1}^{n}(\mathcal{O}_{\lambda(\widetilde{\varepsilon_x})_i}, \acute{\mathcal{E}}).\end{aligned}$$

Therefore, $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$ is a $soft_{Com.}$ set.

The Lindelöf case is analogous, mutatis mutandis.

Corollary 4.4. If $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ is a soft surjective, soft S_p-continuous function and $\widetilde{\mathcal{X}}$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$), then $\widetilde{Y}$ is $soft_{Com.}$ (resp., $soft_{Lind.}$).

Proof. Apply Proposition 4.3 with $(\mathcal{M},\mathcal{E}) = \widetilde{\mathcal{X}}$, using soft surjectivity of $\widetilde{f_{pu}}$.

Corollary 4.5. Under the hypotheses of Corollary 4.4, $\widetilde{Y}$ is soft mildly compact or almost soft compact (resp., soft mildly Lindelöf, or almost soft Lindelöf).

Proof. Combine Corollary 4.4 with Proposition 1.13(1) or Proposition 1.14(1).

Proposition 4.6. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft injective and soft S_p-open. If $\widetilde{f_{pu}}(\widetilde{\mathcal{X}})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set, then $\widetilde{\mathcal{X}}$ is a $soft_{Com.}$ (resp., $soft_{Lind.}$) space.

Proof. Let $\{(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ be a soft open cover of $\widetilde{\mathcal{X}}$. By soft S_p-openness, $\widetilde{f_{pu}}(O_\lambda, \mathcal{E}) \widetilde{\in} \widetilde{S}S_pO(\widetilde{Y})$, $\forall\ \lambda \in \Lambda$. As $\widetilde{f_{pu}}(\widetilde{\mathcal{X}}) \widetilde{\subseteq} \widetilde{f_{pu}}(\widetilde{\cup}_{\lambda\in\Lambda}(O_\lambda, \mathcal{E})) = \widetilde{\cup}_{\lambda\in\Lambda}\widetilde{f_{pu}}(O_\lambda, \mathcal{E})$, then $\{\widetilde{f_{pu}}(O_\lambda, \mathcal{E}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{Y})$-cover of $\widetilde{f_{pu}}(\widetilde{\mathcal{X}})$. The soft S_p-compactness of $\widetilde{f_{pu}}(\widetilde{\mathcal{X}})$, there is $\widetilde{f_{pu}}(O_{\lambda_1}, \mathcal{E}), \widetilde{f_{pu}}(O_{\lambda_2}, \mathcal{E}), \ldots, \widetilde{f_{pu}}(O_{\lambda_n}, \mathcal{E})$ with $\widetilde{f_{pu}}(\widetilde{\mathcal{X}}) \widetilde{\subseteq} \widetilde{\cup}^n_{i=1}\widetilde{f_{pu}}(O_{\lambda_i}, \mathcal{E}) = \widetilde{f_{pu}}(\widetilde{\cup}^n_{i=1}(O_{\lambda_i}, \mathcal{E}))$. By using soft injectivity of $\widetilde{f_{pu}}$, $\widetilde{\mathcal{X}} = \widetilde{f^{-1}_{pu}}(\widetilde{f_{pu}}(\widetilde{\mathcal{X}})) = \widetilde{f^{-1}_{pu}}(\widetilde{f_{pu}}(\widetilde{\cup}^n_{i=1}(O_{\lambda_i}, \mathcal{E}))) = \widetilde{\cup}^n_{i=1}(O_{\lambda_i}, \mathcal{E})$. Thus, $\widetilde{\mathcal{X}}$ is $soft_{com}$.

The Lindelöf case is analogous, mutatis mutandis.

Corollary 4.7. If $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ is soft bijective, soft S_p-open and $\widetilde{Y}$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space, then $\widetilde{\mathcal{X}}$ is a $soft_{Com.}$ (resp., $soft_{Lind.}$) space.

Proof. Apply Proposition 4.6 with $\widetilde{f_{pu}}(\widetilde{\mathcal{X}}) = \widetilde{Y}$, using soft bijectivity of $\widetilde{f_{pu}}$.

Corollary 4.8. Under the hypotheses of Corollary 4.7, then $\widetilde{\mathcal{X}}$ is soft mildly compact or almost soft compact (resp., soft mildly Lindelöf, or almost soft Lindelöf).

Proof. Immediate from Corollary 4.7 combined with Proposition 1.13(1) or Proposition 1.14(1).

Proposition 4.9. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft continuous and soft open. If $(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set, then $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) set.

Proof. Let $\{(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{Y})$-cover of $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$. Since $\widetilde{f_{pu}}$ is soft continuous and soft open, then by Proposition 4.7 of Ref. [27], $\{\widetilde{f^{-1}_{pu}}(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover of $(\mathcal{M},\mathcal{E})$. The soft S_p-compactness of $(\mathcal{M},\mathcal{E})$, there is $\widetilde{f^{-1}_{pu}}(O_{\lambda_1}, \mathcal{E}), \widetilde{f^{-1}_{pu}}(O_{\lambda_2}, \mathcal{E}), \ldots, \widetilde{f^{-1}_{pu}}(O_{\lambda_n}, \mathcal{E})$ with $(\mathcal{M},\mathcal{E}) \widetilde{\subseteq} \widetilde{\cup}^n_{i=1}\ \widetilde{f^{-1}_{pu}}(O_{\lambda_i}, \mathcal{E}) = \widetilde{f^{-1}_{pu}}(\widetilde{\cup}^n_{i=1}(O_{\lambda_i}, \mathcal{E}))$. Thus, we get $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E}) \widetilde{\subseteq} \widetilde{f_{pu}}(\widetilde{f^{-1}_{pu}}(\widetilde{\cup}^n_{i=1}(O_{\lambda_i}, \mathcal{E}))) \widetilde{\subseteq} \widetilde{\cup}^n_{i=1}(O_{\lambda_i}, \mathcal{E})$. This shows that $\widetilde{f_{pu}}(\mathcal{M},\mathcal{E})$ is a $soft_{S_p-Com}$ set.

The Lindelöf case is analogous, mutatis mutandis.

Proposition 4.10. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft surjective and soft irresolute. If $\widetilde{\mathcal{X}}$ is a $soft_{semi-Com.}$ (resp., $soft_{semi-Lind.}$) space, then $\widetilde{Y}$ is a $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$) space.

Proof. Every $\widetilde{S}S_pO(\widetilde{Y})$-cover is an $\widetilde{S}SO(\widetilde{Y})$-cover, its preimage under the soft irresolute $\widetilde{f_{pu}}$ is a $\widetilde{S}SO(\widetilde{\mathcal{X}})$-cover. The soft semi-compactness (resp., soft semi-Lindelöfness) of $\widetilde{\mathcal{X}}$ extracts a finite (resp., countable) subcover, and soft surjectivity transports it to $\widetilde{Y}$.

Proposition 4.11. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft bijective and soft irresolute open. If $\widetilde{Y}$ is $soft_{semi-Com.}$ (resp., $soft_{semi-Lind.}$), then $\widetilde{\mathcal{X}}$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proof. Every $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover is an $\widetilde{S}SO(\widetilde{\mathcal{X}})$-cover, its image under the soft irresolute open $\widetilde{f_{pu}}$ is a $\widetilde{S}SO(\widetilde{Y})$-cover. The soft semi-compactness (resp., soft semi-Lindelöfness) of $\widetilde{Y}$ yields a finite (resp., countable) subcover, and soft bijectivity transports it back to $\widetilde{\mathcal{X}}$.

Proposition 4.12. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft surjective and soft S_p-irresolute. If $\widetilde{\mathcal{X}}$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$), then $\widetilde{Y}$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$).

Proof. Let $\{(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ be an $\widetilde{S}S_pO(\widetilde{Y})$-cover. By soft S_p-irresoluteness, $\{\widetilde{f^{-1}_{pu}}(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. The soft S_p-compactness (resp., soft S_p-Lindelöfness) of $\widetilde{\mathcal{X}}$ yields a finite (resp., countable) subcover, and soft surjectivity of $\widetilde{f_{pu}}$ transports back to $\widetilde{Y}$.

Definition 4.13. A soft function $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ is called a:

1. soft almost S_p-continuous iff $\widetilde{f^{-1}_{pu}}(B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{S}S_pO(\widetilde{\mathcal{X}})$, $\forall (B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{S}RO(\widetilde{Y})$.
2. soft weakly S_p-continuous at $\widetilde{\varepsilon_x} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})$, if $\forall\ (B, \acute{\mathcal{E}}) \widetilde{\in} \widetilde{\sigma}$ containing $\widetilde{f_{pu}}(\widetilde{\varepsilon_x})$ there is $\widetilde{\varepsilon_x} \widetilde{\in} (\mathcal{M}, \mathcal{E}) \widetilde{\in} \widetilde{S}S_pO(\widetilde{\mathcal{X}})$ with $\widetilde{f_{pu}}(\mathcal{M}, \mathcal{E}) \widetilde{\subseteq} \widetilde{scl}(B, \acute{\mathcal{E}})$. $\widetilde{f_{pu}}$ is soft weakly S_p-continuous, if this hold at every soft point of $\widetilde{\mathcal{X}}$.

Proposition 4.14. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft surjective and soft almost S_p-continuous. If $\widetilde{\mathcal{X}}$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$), then $\widetilde{Y}$ is $soft_{N.\ Com.}$ (resp., $soft_{N.\ Lind.}$).

Proof. Let $\{(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ be an $\widetilde{S}RO(\widetilde{Y})$-cover. By Definition 4.13(1), $\{\widetilde{f^{-1}_{pu}}(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. The soft S_p-compactness (resp., soft S_p-Lindelöfness) of $\widetilde{\mathcal{X}}$ yields a finite (resp., countable) subcover, and soft surjectivity transports it back to $\widetilde{Y}$.

Proposition 4.15. Let $\widetilde{f_{pu}} : (\widetilde{\mathcal{X}}, \widetilde{\mathcal{T}}, \mathcal{E}) \to (\widetilde{Y}, \widetilde{\sigma}, \acute{\mathcal{E}})$ be soft surjective and soft weakly S_p-continuous. If $\widetilde{\mathcal{X}}$ is $soft_{S_p-Com.}$ (resp., $soft_{S_p-Lind.}$), then $\widetilde{Y}$ is almost soft compact (resp., almost soft Lindelöf).

Proof. Let $\{(O_\lambda, \acute{\mathcal{E}}) : \lambda \in \Lambda\}$ be a soft open cover of $\widetilde{Y}$. Then, $\forall\ \widetilde{\varepsilon_x} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})$, there is $\lambda(\widetilde{\varepsilon_x}) \in \Lambda$ with $\widetilde{f_{pu}}(\widetilde{\varepsilon_x}) \widetilde{\in} (O_{\lambda(\widetilde{\varepsilon_x})}, \acute{\mathcal{E}})$. By Definition 4.13(2), there exists $(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E}) \widetilde{\in} \widetilde{S}S_pO(\widetilde{\mathcal{X}})$ containing $\widetilde{\varepsilon_x}$ with $\widetilde{f_{pu}}(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E}) \widetilde{\subseteq} \widetilde{scl}(O_{\lambda(\widetilde{\varepsilon_x})}, \acute{\mathcal{E}})$. Now, we have $\{(B_{\lambda(\widetilde{\varepsilon_x})}, \mathcal{E}) : \widetilde{\varepsilon_x} \widetilde{\in} \widetilde{S}P(\widetilde{\mathcal{X}})\}$ is a $\widetilde{S}S_pO(\widetilde{\mathcal{X}})$-cover. The soft S_p-compactness of $\widetilde{\mathcal{X}}$ yields a finite soft subcollection with $\widetilde{\mathcal{X}} = \widetilde{\cup}^n_{i=1}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})$. By applying the soft surjectivity of $\widetilde{f_{pu}}$, $\widetilde{Y} = \widetilde{f_{pu}}(\widetilde{\mathcal{X}}) = \widetilde{f_{pu}}(\widetilde{\cup}^n_{i=1}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})) = \widetilde{\cup}^n_{i=1}(\widetilde{f_{pu}}(B_{\lambda(\widetilde{\varepsilon_x})_i}, \mathcal{E})) \widetilde{\subseteq} \widetilde{\cup}^n_{i=1}\ \widetilde{scl}(O_{\lambda(\widetilde{\varepsilon_x})_i}, \acute{\mathcal{E}})$. Thus, $\widetilde{Y}$ is almost soft compact. The Lindelöf case is analogous, mutatis mutandis.

5. Conclusion

In this paper, we have introduced and systematically studied two new compactness-type notions in soft topology, namely soft S_p-compact and soft S_p-Lindelöf spaces, by utilizing the

class of soft S_p-open sets as a covering tool. Specifically, soft semi-compactness implies soft S_p-compactness, which in turn implies mildly soft compactness, thereby occupying a well-defined position in the hierarchy of soft compactness-type properties and filling a natural gap in the existing literature.

Several characterizations of soft S_p-compact and soft S_p-Lindelöf spaces were established. In particular, a characterization via the finite (respectively, countable) soft intersection property of families of soft S_p-closed sets was obtained, extending a classical approach to this new framework. Under additional structural conditions, these relations can be further clarified:

In soft locally indiscrete spaces, soft S_p-compactness is equivalent to soft compactness, soft semi-compactness, and soft α-compactness. In addition, in soft T_1-spaces, soft S_p-compactness is equivalent to soft semi-compactness and implies both soft compactness and soft α-compactness.

In soft extremally disconnected spaces, both soft α-compactness and soft pre-compactness imply soft S_p-compactness, which in turn implies soft nearly compactness. Furthermore, soft S_p-compactness implies soft$_{\text{int}}$ compactness in soft extremally disconnected spaces, while the converse implication holds under soft p-Alexandroffness.

The notions of soft S_p-compact and soft S_p-Lindelöf sets and subspaces were introduced, and their stability under soft S_p-closed subsets, soft clopen subsets, and finite unions was investigated. The invariance and preservation of these properties were fully analysed under several types of soft mappings, including soft S_p-continuous, soft S_p-irresolute, soft almost S_p-continuous, and soft weakly S_p-continuous functions. The concepts introduced here are proper generalizations of existing compactness-type notions, as confirmed by illustrative counterexamples throughout the paper.

This work provides a foundation for further investigations within the framework of soft topology, in particular the study of soft S_p-paracompactness, soft S_p-connectedness, and soft S_p-separation axioms.

Data availability

No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Declaration of competing interest

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