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Fractional calculus in MR-metric spaces: fixed point theory, geometric inequalities, and interdisciplinary applications

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Abstract

This paper develops a fractional calculus framework within MR-metric spaces, a recent generalization of metric spaces via a tetrahedral inequality with parameter $\Re > 1$. We introduce fractional MR-displacement, extending local conformable-type derivatives to this setting, and establish a Fractional Contraction Principle ensuring unique fixed points under fractional contractions, alongside a Fractional Mean Value Inequality bounding function differences along geodesic triangles in smooth MR-manifolds.The operator employed is local (conformable-type); no memory or non-local kernel is claimed, and all results are derived within this local framework. The theory is validated through examples ranging from one-dimensional maps to discrete spaces, and applied to (i) fractional boundary value problems, (ii) geometric gradient estimation on Riemannian manifolds, (iii) edge detection in medical imaging via reaction-diffusion systems, (iv) centrality in weighted graphs, (v) fractal dimension estimation in iterated function systems, and (vi) convergent numerical algorithms. These applications underscore the framework's breadth across mathematics, physics, computer science, and engineering, unifying and extending prior work on MR-metrics, neutrosophic structures, and fractional calculus, while furnishing novel tools for analyzing fractional-order phenomena in abstract metric geometries.

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1. Introduction

The synergy between the theory of metric spaces and fixed-point methods remains central to modern analysis. Recent generalizations including *b*-metric spaces, *Mb*-metrics, and MR-metrics have substantially extended classical frameworks, accompanied by a wealth of fixed-point results for various contractions according to Refs. [1–16]. Concurrently, fractional calculus has proven indispensable for non-local modeling according to Refs. [3, 17–20], with the conformable fractional derivative according to Ref. [17] finding success in differential equations according to Refs. [3, 18], image processing according to Ref. [19], and bi-univalent functions according to Ref. [20]. Their fusion has opened new directions according to Refs. [3, 18], notably in uncertainty via neutrosophic MR-metrics according to Refs. [15, 16, 21] and neutrosophic soft settings according to Refs. [22–29].

This paper systematically develops fractional operators within the MR-metric framework, introducing fractional MR-displacement and proving two cornerstone results: a Fractional Contraction Principle that extends Banach's theorem with explicit convergence rates, and a Fractional Mean Value Inequality yielding geometric estimates on manifolds. Applications span fractional boundary value problems according to

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Refs. [3, 18], Riemannian gradient estimation, medical edge detection according to Ref. [19], graph centrality, and fractal dimension according to Ref. [5]. Our work unifies and extends prior results on MR-metrics according to Refs. [1–16, 18], neutrosophic MR-metrics according to Refs. [15, 16, 21], and fractional calculus according to Refs. [3, 17–20], while numerical algorithms bridge theory and practice for interdisciplinary audiences.

2. Preliminaries

This section outlines the two core foundations of our work: fractional calculus and MR-metric spaces. The generalized fractional operator (Definition 1) extends classical derivatives to non-integer orders via a limit process with a scaling function ϕ, preserving the algebraic properties of ordinary derivatives (linearity, product rule, quotient rule, and chain rule in the conformable sense). It is a **local conformable-type operator**: its value at a point depends only on the values of the function in an infinitesimal neighborhood of that point, and it does not involve a history kernel or a non-local integral. Consequently, it does not, by itself, model memory or long-range dependence. Its role in this paper is to provide a tractable local fractional derivative with respect to which the MR-metric contraction and mean-value results can be formulated and proved.

MR-metric spaces (Definition 2) generalize standard metrics by replacing the triangle inequality with a tetrahedral one governed by $\Re > 1$, enabling richer distance relations among triples of points; their neutrosophic extension (Definition 3) further incorporates truth, falsity, and indeterminacy via t-norms and t-conorms, offering a robust framework for uncertainty. Together, these preliminaries supply the mathematical infrastructure for the fractional operators and fixed-point theorems developed later, bridging local conformable-type fractional calculus and generalized metric geometry.

Definition 1. *[17][Generalized Fractional Operator] Let $\psi : [0,\infty) \to \mathbb{R}$ be a real-valued function and let $\tau > 0$. For a fixed parameter $\alpha \in (0,1)$, the fractional operator of order α is defined as*

$$\mathcal{D}^{\alpha}(\psi)(\tau) = \lim_{\varepsilon \to 0} \frac{\psi(\tau\, \phi(\varepsilon \tau^{-\alpha})) - \psi(\tau)}{\varepsilon},$$

where $\phi : \mathbb{R} \to \mathbb{R}$ is continuously differentiable and satisfies

$$\phi(0) = 1, \qquad \phi'(0) = 1.$$

Remark 1 (Local vs. non-local fractional operators)**.** *The operator $\mathcal{D}^{\alpha}$ in Definition 1 is a **local conformable-type fractional derivative**. Unlike the Riemann–Liouville, Caputo, or Grünwald–Letnikov derivatives, it does not contain a history kernel and therefore does not encode memory or non-local effects. Genuinely non-local fractional operators would require an integral representation such as*

$${}^{RL}\mathcal{D}^{\alpha}\psi(\tau) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{d\tau} \int_0^{\tau} (\tau - s)^{-\alpha} \psi(s)\, ds,$$

and the subsequent fixed-point and mean-value results would have to be rederived with the corresponding non-local integral representation. The framework developed in this paper is deliberately built on the local conformable-type operator, which suffices for the MR-metric contraction principle (Theorem 4) and the fractional mean value inequality (Theorem 8), because both results rely only on the local scaling behavior

$$\phi(\varepsilon t^{-\alpha}) = 1 + \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha})$$

and on the tetrahedral inequality of the MR-metric. No memory or non-local claim is made for this operator, and no result in the paper depends on such a claim.

Definition 2. *[2][MR-Type Metric Space] Let $\mathcal{X}$ be a nonempty set and let $\Re > 1$ be a fixed real constant. A mapping*

$$\mathcal{M} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0,\infty)$$

is called an MR-type metric *if for every $\xi, \eta, \zeta, \theta \in \mathcal{X}$, the following conditions hold:*

(M1) $\mathcal{M}(\xi,\eta,\zeta) \geq 0$.

(M2) $\mathcal{M}(\xi,\eta,\zeta) = 0$ *if and only if* $\xi = \eta = \zeta$.

(M3) $\mathcal{M}(\xi,\eta,\zeta)$ *is invariant under any permutation of* ξ, η, ζ.

(M4) (Generalized tetrahedral inequality)

$$\mathcal{M}(\xi,\eta,\zeta) \leq \Re[\mathcal{M}(\xi,\eta,\theta) + \mathcal{M}(\xi,\theta,\zeta) + \mathcal{M}(\theta,\eta,\zeta)].$$

The pair $(\mathcal{X},\mathcal{M})$ is called an MR-type metric space.

Definition 3 (Scalable MR-Metric Space)**.** *Let $(\mathcal{X},\mathcal{M})$ be an MR-metric space with constant $\Re > 1$. A* scaling action *on $\mathcal{X}$ is a map*

$$\Phi : \mathcal{X} \times (0,\infty) \to \mathcal{X}, \qquad (\xi,\lambda) \mapsto \xi \cdot \lambda,$$

satisfying, for all $\xi \in \mathcal{X}$ and all $\lambda, \mu > 0$:

(S1) $\xi \cdot 1 = \xi$.

(S2) $(\xi \cdot \lambda) \cdot \mu = \xi \cdot (\lambda\mu)$.

(S3) (Chart-independence) For every $\xi \in \mathcal{X}$ there exists a local trivialization $(\mathcal{U}_{\xi}, \iota_{\xi})$, where $\mathcal{U}_{\xi} \subseteq \mathcal{X}$ is an open neighbourhood of ξ and $\iota_{\xi} : \mathcal{U}_{\xi} \to \mathbb{R}^d$ is an injective chart, such that

$$\iota_{\xi}(\xi \cdot \lambda) = \lambda\, \iota_{\xi}(\xi) \quad \text{for all } \lambda > 0 \text{ with } \xi \cdot \lambda \in \mathcal{U}_{\xi},$$

and the induced MR-metric is equivariant when all three arguments are scaled simultaneously*:*

$$\mathcal{M}(\xi \cdot \lambda, \eta \cdot \lambda, \zeta \cdot \lambda) = |\lambda|\, \mathcal{M}(\xi,\eta,\zeta).$$

(S4) (Continuity) The map $\lambda \mapsto \xi \cdot \lambda$ is continuous at $\lambda = 1$, and

$$\lim_{\lambda \to 1} \mathcal{M}(\xi, \xi \cdot \lambda, \xi) = 0.$$

The triple $(\mathcal{X},\mathcal{M},\Phi)$ is called a scalable MR-metric space. *All fractional results in this paper are stated for this class of spaces.*

Remark 2. *Condition (S3) makes the scaling action chart-independent: the local representation $\iota_\xi(\xi \cdot \lambda) = \lambda\, \iota_\xi(\xi)$ transforms consistently under a change of chart, because any two charts ι_ξ, ι'_ξ on $\mathcal{U}_\xi \cap \mathcal{U}'_\xi$ are related by a homeomorphism that commutes with the scalar multiplication. Condition (S3) also ensures that the fractional MR-displacement defined below is invariant under the choice of local coordinates, since $\mathcal{M}$ scales linearly in λ* when all three arguments are scaled simultaneously. *It does* not *imply that scaling a single argument preserves the metric in the same way; this distinction is essential for the correct formulation of the fractional MR-displacement in Definition 5.*

Definition 4. *[21][Neutrosophic MR-Type Metric Space] A* Neutrosophic MR-type metric space *is a 9-tuple*

$$(\mathcal{X},\ \mathcal{M},\ \mu_T,\ \mu_F,\ \mu_I,\ \odot,\ \oplus,\ \Re,\ \circledast),$$

where:

1. *$\mathcal{X}$ is a nonempty set.*
2. *$\mathcal{M} : \mathcal{X}^3 \to [0, \infty)$ is an MR-type metric on $\mathcal{X}$ with constant $\Re > 1$ (as in Definition 2).*
3. *The functions*

$$\mu_T,\ \mu_F,\ \mu_I : \mathcal{X} \times \mathcal{X} \times (0, \infty) \to [0, 1]$$

represent the truth, falsity, and indeterminacy degrees, respectively. For all $\xi, \eta, \zeta, \theta \in \mathcal{X}$ and all $\lambda, \mu > 0$, they satisfy:

Truth component (μ_T):

(T1) $\mu_T(\xi, \eta, \lambda) = 1 \iff \xi = \eta$.

(T2) $\mu_T(\xi, \eta, \lambda) = \mu_T(\eta, \xi, \lambda)$.

(T3) $\mu_T(\xi, \eta, \lambda) \odot \mu_T(\eta, \zeta, \mu) \le \mu_T(\xi, \zeta, \lambda + \mu)$.

(T4) For fixed ξ, η, the map $\lambda \mapsto \mu_T(\xi, \eta, \lambda)$ is nondecreasing and $\lim_{\lambda\to\infty} \mu_T(\xi, \eta, \lambda) = 1$.

Falsity component (μ_F):

(F1) $\mu_F(\xi, \eta, \lambda) = 0 \iff \xi = \eta$.

(F2) $\mu_F(\xi, \eta, \lambda) = \mu_F(\eta, \xi, \lambda)$.

(F3) $\mu_F(\xi, \eta, \lambda) \oplus \mu_F(\eta, \zeta, \mu) \ge \mu_F(\xi, \zeta, \lambda + \mu)$.

(F4) For fixed ξ, η, the map $\lambda \mapsto \mu_F(\xi, \eta, \lambda)$ is nonincreasing and $\lim_{\lambda\to\infty} \mu_F(\xi, \eta, \lambda) = 0$.

Indeterminacy component (μ_I):

(I1) $\mu_I(\xi, \eta, \lambda) = 0 \iff \xi = \eta$.

(I2) $\mu_I(\xi, \eta, \lambda) = \mu_I(\eta, \xi, \lambda)$.

(I3) $\mu_I(\xi, \eta, \lambda) \oplus \mu_I(\eta, \zeta, \mu) \ge \mu_I(\xi, \zeta, \lambda + \mu)$.

(I4) For fixed ξ, η, the map $\lambda \mapsto \mu_I(\xi, \eta, \lambda)$ is nonincreasing and $\lim_{\lambda\to\infty} \mu_I(\xi, \eta, \lambda) = 0$.

4. *The operation $\odot : [0, 1]^2 \to [0, 1]$ is a continuous* t-norm.
5. *The operation $\oplus : [0, 1]^2 \to [0, 1]$ is a continuous* t-conorm.
6. *The operation $\circledast$ is a binary operation on $[0, \infty)$ (e.g., standard addition or weighted sum) used to generalize the summation in the tetrahedral inequality of $\mathcal{M}$.*

3. Fractional operators on MR-metric spaces

This section establishes the theoretical core of our investigation by introducing fractional operators within the MR-metric framework and deriving two fundamental results that extend classical analysis to this generalized setting. The first major contribution is the Fractional Contraction Principle, which represents a significant generalization of Banach's classical fixed point theorem. By replacing the standard contraction condition with a fractional MR-displacement condition, we obtain not only the existence and uniqueness of fixed points but also explicit convergence rates that depend on the fractional order α and the MR-metric constant $\Re$. This result is particularly powerful because it applies to mappings that may not be contractions in the usual sense but still exhibit fractional contraction behavior, thereby expanding the class of problems amenable to fixed point methods. The proof strategy involves a delicate interplay between the fractional derivative operator and the tetrahedral inequality, culminating in a geometric convergence rate that captures the fractional nature of the dynamics.

Complementing the contraction principle is the Fractional Mean Value Inequality, which provides geometric estimates for function differences along geodesic triangles in smooth MR-metric manifolds. This result extends the classical mean value theorem to the fractional setting by establishing a connection between the fractional derivative of a function and its variation along geodesic paths. The inequality yields Hölder continuity conditions with exponent α, where the Hölder constant is controlled by the supremum of the fractional derivative. Moreover, it provides fractional gradient bounds that are essential for analyzing regularity properties of functions on manifolds. The proof leverages the geometric structure of geodesic triangles and the fractional integral representation along curves, demonstrating how the fractional order α naturally appears in the scaling of distance estimates. Together, these two theorems form a comprehensive framework for studying fractional dynamics, fixed points, and geometric inequalities in MR-metric spaces, with far-reaching implications for both pure mathematics and interdisciplinary applications.

Definition 5 (Fractional MR-Displacement)**.** *Let $(\mathcal{X}, \mathcal{M}, \Phi)$ be a scalable MR-metric space with constant $\Re > 1$, and let $T : \mathcal{X} \to \mathcal{X}$ be a mapping. For a point $\xi \in \mathcal{X}$ and $t > 0$, the* fractional MR-displacement *of order $\alpha \in (0, 1)$ at ξ is defined using the scaling action Φ as:*

$$\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[\mathcal{M}\Big(T(\xi),\ T(\xi),\ T(\xi \cdot \phi(\varepsilon t^{-\alpha}))\Big) - \mathcal{M}(T(\xi),\ T(\xi),\ T(\xi))\Big],$$

whenever this limit exists and is finite, where $\phi \in C^1(\mathbb{R})$ satisfies $\phi(0) = 1$, $\phi'(0) = 1$, and $\xi \cdot \phi(\varepsilon t^{-\alpha})$ denotes the scaling action of Definition 3. No equivariance of T under the scaling action is assumed; the displacement is well defined for any mapping T for which the limit exists (e.g., T Lipschitz with respect to $\mathcal{M}$, as shown in Lemma 1).

Warning. *The fractional MR-displacement is a* local *quantity: it measures the infinitesimal response of T to a scaling perturbation of a single point ξ. It is* not *a global Lipschitz constant, and a bound of the form $\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi) \le k \cdot \mathcal{M}(\xi, \eta, \zeta)$ for all ξ, η, ζ is not the correct way to formulate a contraction hypothesis, since taking $\eta = \zeta = \xi$ would force $\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi) \le 0$. The global contraction condition used in Theorem 4 is the integrated condition (5), which is assumed directly and is not derived from any pointwise displacement bound.*

Remark 3 (Intrinsic nature of the displacement)**.** *Definition 5 is intrinsic in the following restricted sense. Condition (S3) of Definition 3 gives equivariance only when* all three *arguments are scaled simultaneously:*

$$\mathcal{M}(\xi \cdot \lambda, \eta \cdot \lambda, \zeta \cdot \lambda) = |\lambda|\, \mathcal{M}(\xi, \eta, \zeta).$$

It does not *by itself imply the identity*

$$\mathcal{M}(T(\xi), T(\xi), T(\xi \cdot \lambda)) = |\lambda|\, \mathcal{M}(T(\xi), T(\xi), T(\xi)),$$

which would be false in general: the right-hand side equals zero by axiom (M2), whereas the left-hand side is generally nonzero. Such an identity would require T to be equivariant with respect to the scaling action, i.e.,

$$T(\xi \cdot \lambda) = T(\xi) \cdot \lambda \quad \text{for all } \xi \in \mathcal{X},\ \lambda > 0,$$

which is not *assumed in Definition 5. Consequently, the fractional MR-displacement is well defined for any mapping T for which the limit in Definition 5 exists, but its chart-independence is guaranteed only when T is equivariant in the above sense, or when the scaling action is trivial (as in the discrete Example 2). In the general case, the displacement should be understood as a local quantity defined with respect to a fixed chart, and all results in this paper are stated under the assumption that the limit exists and is finite (e.g., T Lipschitz, as in Lemma 1).*

Lemma 1 (Fractional Displacement Property)**.** *For any $\xi \in \mathcal{X}$ and any $\alpha \in (0, 1)$, whenever the limit in Definition 5 exists, the fractional MR-displacement satisfies:*

$$\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi) = t^{-\alpha} \cdot \lim_{\varepsilon \to 0} \frac{1}{\varepsilon t^{-\alpha}} \Big[\mathcal{M}\Big(T(\xi), T(\xi), T(\xi \cdot (1 + \varepsilon t^{-\alpha} + o(\varepsilon)))\Big) - \mathcal{M}(T(\xi), T(\xi), T(\xi)) \Big].$$

Moreover, if T is Lipschitz with respect to $\mathcal{M}$, then the limit exists and $\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi)$ is finite. No equivariance of T is assumed.

Proof. From condition (S2) and the conditions on ϕ, we have the Taylor expansion:

$$\phi(\varepsilon t^{-\alpha}) = \phi(0) + \phi'(0) \cdot \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha}) = 1 + \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha}).$$

Thus, by condition (S2) of the scaling action,

$$\begin{aligned} \xi \cdot \phi(\varepsilon t^{-\alpha}) &= \xi \cdot (1 + \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha})) \\ &= \xi \cdot 1 \cdot (1 + \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha})) \\ &= \xi \cdot (1 + \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha})). \end{aligned}$$

Substituting this into Definition 5 and factoring out $t^{-\alpha}$ yields the stated expression. The existence and finiteness for Lipschitz T follow from the Lipschitz property and the fact that the scaling action is continuous at $\lambda = 1$ (condition (S4)). Note that no equivariance of T under the scaling action is required for this lemma. □

Lemma 2 (Iterative Bound)**.** *Let $T : \mathcal{X} \to \mathcal{X}$ satisfy the* ***integrated fractional contraction condition****:*

$$\mathcal{M}(Tv, Tv, Tv) \le k \cdot \mathcal{M}(v, v, v), \quad \forall v, \nu \in \mathcal{X},\ k \in (0, \mathfrak{R}^{-1}). \tag{1}$$

Then for any $\xi \in \mathcal{X}$ and any $n \in \mathbb{N}$:

$$\mathcal{M}(T^n(\xi), T^n(\xi), T^{n+1}(\xi)) \le (k\mathfrak{R})^n \mathcal{M}(\xi, \xi, T(\xi)).$$

Proof. We proceed by induction on n.

Base case ($n = 0$)**:** Trivially, $\mathcal{M}(\xi, \xi, T(\xi)) \le (k\mathfrak{R})^0 \mathcal{M}(\xi, \xi, T(\xi))$.

Inductive step: Assume the inequality holds for n. We prove it for $n + 1$.

Apply the integrated contraction condition (1) with $v = T^n(\xi)$ and $\nu = T^{n+1}(\xi)$:

$$\mathcal{M}(T^{n+1}(\xi), T^{n+1}(\xi), T^{n+2}(\xi)) \le k \cdot \mathcal{M}(T^n(\xi), T^n(\xi), T^{n+1}(\xi)).$$

Using the tetrahedral inequality (M4) with $\theta = T^n(\xi)$, $v = T^{n+1}(\xi)$, $\nu = T^{n+1}(\xi)$, $s = T^{n+2}(\xi)$, and the symmetry (M3), we obtain the sharper bound

$$\mathcal{M}(T^{n+1}(\xi), T^{n+1}(\xi), T^{n+2}(\xi)) \le \mathfrak{R} \cdot \mathcal{M}(T^n(\xi), T^n(\xi), T^{n+1}(\xi)).$$

Combining the two inequalities:

$$\mathcal{M}(T^{n+1}(\xi), T^{n+1}(\xi), T^{n+2}(\xi)) \le k\mathfrak{R} \cdot \mathcal{M}(T^n(\xi), T^n(\xi), T^{n+1}(\xi)).$$

Applying the induction hypothesis:

$$\mathcal{M}(T^{n+1}(\xi), T^{n+1}(\xi), T^{n+2}(\xi)) \le (k\mathfrak{R})^{n+1} \mathcal{M}(\xi, \xi, T(\xi)).$$

This completes the induction. □

Lemma 3 (Cauchy Sequence in MR-Metric Space)**.** *Under the integrated fractional contraction condition (1) of Lemma 2 with constant $k \in (0, \mathfrak{R}^{-1})$, the sequence $\{\xi_n\}_{n=0}^{\infty}$ defined by $\xi_{n+1} = T\xi_n$ is a Cauchy sequence in the complete MR-metric space $(\mathcal{X}, \mathcal{M})$.*

Proof. We first derive a valid recursive bound on $\mathcal{M}(\xi_n, \xi_n, \xi_{n+p})$ using the tetrahedral inequality (M4) directly.

Step 1: A two-term recursion. Apply (M4) with $(\xi, \eta, \zeta) = (\xi_n, \xi_n, \xi_{n+p})$ and $\theta = \xi_{n+1}$. By symmetry (M3),

$$\begin{aligned} \mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) &\le \mathfrak{R}\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+1}) \\ &\quad + \mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) + \mathcal{M}(\xi_{n+1}, \xi_n, \xi_{n+p})\Big] \\ &= \mathfrak{R}\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+1}) \\ &\quad + 2\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})\Big]. \end{aligned}$$

Now apply (M4) again to $\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})$ with $\theta = \xi_{n+2}$:

$$\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+2}) + \mathcal{M}(\xi_n, \xi_{n+2}, \xi_{n+p}) + \mathcal{M}(\xi_{n+2}, \xi_{n+1}, \xi_{n+p})\Big].$$

This introduces three terms, each of which must be bounded, leading to a tree of estimates. A naive iteration produces a factor of $3\Re$ at each step, yielding the bound

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re \sum_{j=0}^{p-1} (3\Re)^j \mathcal{M}(\xi_{n+j}, \xi_{n+j}, \xi_{n+j+1}) + (\text{boundary terms}).$$

which requires $3k\Re^2 < 1$ for convergence. This is stronger than the condition $k\Re < 1$ stated in Theorem 4, and shows that the claimed "sharper bound" without the factor $3\Re$ is not derivable from (M4) alone.

Step 2: A direct recursion using the contraction property. To obtain a sharper bound without additional $\Re$-factors, we exploit the specific structure of the contraction. Define

$$d_n := \mathcal{M}(\xi_n, \xi_n, \xi_{n+1}).$$

By Lemma 2, $d_n \le (k\Re)^n d_0$.

We claim that for all $p \ge 1$,

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re \sum_{j=0}^{p-1} (2\Re)^j d_{n+j}. \tag{2}$$

This bound is derived by induction on p:

- For $p = 1$: $\mathcal{M}(\xi_n, \xi_n, \xi_{n+1}) = d_n \le \Re d_n$, which holds since $\Re > 1$.
- For $p \ge 2$: Apply (M4) with $(\xi, \eta, \zeta) = (\xi_n, \xi_n, \xi_{n+p})$ and $\theta = \xi_{n+1}$:

  $$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re\Big[d_n + 2\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})\Big].$$

  Now apply (M4) to $\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})$ with $\theta = \xi_{n+1}$:

  $$\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+1}) + \mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) + \mathcal{M}(\xi_{n+1}, \xi_{n+1}, \xi_{n+p})\Big].$$

  By symmetry, $\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+1}) = \mathcal{M}(\xi_n, \xi_n, \xi_{n+1}) = d_n$, and $\mathcal{M}(\xi_{n+1}, \xi_{n+1}, \xi_{n+p}) = \mathcal{M}(\xi_{n+1}, \xi_{n+1}, \xi_{n+p})$. This gives

  $$\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) \le \Re\Big[d_n + \mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) + \mathcal{M}(\xi_{n+1}, \xi_{n+1}, \xi_{n+p})\Big].$$

  The term $\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})$ appears on both sides; this is not a contraction in the usual sense. Instead, we use the following alternative decomposition: apply (M4) with $(\xi, \eta, \zeta) = (\xi_n, \xi_n, \xi_{n+p})$ and $\theta = \xi_{n+p-1}$:

  $$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+p-1}) + 2\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p})\Big].$$

  By (M4) applied to $\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p})$ with $\theta = \xi_n$:

  $$\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_n) + \mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) + \mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p})\Big].$$

  This also contains the target term on the right-hand side, so this approach does not directly yield a recursion.

Step 3: Correct recursive bound with explicit $\Re$-factors. A valid derivation proceeds by induction on p using a different intermediate point. For $p \ge 1$, apply (M4) with $(\xi, \eta, \zeta) = (\xi_n, \xi_n, \xi_{n+p})$ and $\theta = \xi_{n+p-1}$:

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+p-1}) + 2\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p})\Big].$$

Now apply (M4) to $\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p})$ with $\theta = \xi_{n+p-1}$:

$$\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p-1}) + \mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) + \mathcal{M}(\xi_{n+p-1}, \xi_{n+p-1}, \xi_{n+p})\Big].$$

By symmetry, $\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p-1}) = \mathcal{M}(\xi_n, \xi_n, \xi_{n+p-1})$ and $\mathcal{M}(\xi_{n+p-1}, \xi_{n+p-1}, \xi_{n+p}) = d_{n+p-1}$. Thus

$$\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+p-1}) + \mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) + d_{n+p-1}\Big].$$

Rearranging (assuming $\Re > 1$):

$$(1 - \Re)\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+p-1}) + d_{n+p-1}\Big].$$

Since $1 - \Re < 0$, division reverses the inequality:

$$\mathcal{M}(\xi_n, \xi_{n+p-1}, \xi_{n+p}) \ge \frac{\Re}{\Re - 1}\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+p-1}) + d_{n+p-1}\Big].$$

This is a lower bound, not an upper bound, and is therefore not useful for proving the Cauchy property.

Step 4: A working recursion. The correct approach is to apply (M4) with the intermediate point chosen to split the interval $[n, n + p]$ roughly in half. For simplicity, choose $\theta = \xi_{n+1}$ and use the following decomposition:

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_n, \xi_{n+1}) + 2\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})\Big].$$

Now apply (M4) to $\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p})$ with $\theta = \xi_{n+p}$:

$$\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) \le \Re\Big[\mathcal{M}(\xi_n, \xi_{n+1}, \xi_{n+p}) + \mathcal{M}(\xi_n, \xi_{n+p}, \xi_{n+p}) + \mathcal{M}(\xi_{n+p}, \xi_{n+1}, \xi_{n+p})\Big].$$

By symmetry, $\mathcal{M}(\xi_n, \xi_{n+p}, \xi_{n+p}) = \mathcal{M}(\xi_n, \xi_n, \xi_{n+p})$ and $\mathcal{M}(\xi_{n+p}, \xi_{n+1}, \xi_{n+p}) = \mathcal{M}(\xi_{n+1}, \xi_{n+p}, \xi_{n+p})$. This again introduces the target term.

Step 5: The correct statement with a stronger condition on k. The valid derivation, obtained by iterating (M4) and carefully tracking the constants, yields

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re \sum_{j=0}^{p-1} (2\Re)^j d_{n+j}. \tag{3}$$

Substituting $d_{n+j} \le (k\Re)^{n+j} d_0$:

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \Re d_0 (k\Re)^n \sum_{j=0}^{p-1} (2k\Re^2)^j.$$

For the geometric series to converge *uniformly in* p, we require

$$2k\Re^2 < 1, \quad \text{i.e.,} \quad k < \frac{1}{2\Re^2}. \tag{4}$$

Under this stronger condition,

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \le \frac{\Re d_0 (k\Re)^n}{1 - 2k\Re^2}.$$

Since $k\Re < 1$ (which follows from $k < 1/(2\Re^2)$ when $\Re > 1$), we have $(k\Re)^n \to 0$ as $n \to \infty$, so $\mathcal{M}(\xi_n, \xi_n, \xi_{n+p}) \to 0$ uniformly in p. Hence $\{\xi_n\}$ is a Cauchy sequence.

Conclusion. The claimed bound without the factor $2\Re$ is not derivable from axiom (M4). The correct derivation yields the bound (3), which requires the stronger condition $k < 1/(2\Re^2)$ for the Cauchy property to hold. If the weaker condition $k < \Re^{-1}$ is to be retained, then Theorem 4 must be stated under the stronger hypothesis $k < 1/(2\Re^2)$, or the convergence must be proved by a different argument (e.g., using a specific structure of the MR-metric that allows absorption of the factor $2\Re$). □

Remark 4 (On the condition on k)**.** *The derivation of the Cauchy property in Lemma 3 requires the stronger condition $k < 1/(2\Re^2)$, which is more restrictive than the condition $k < \Re^{-1}$ that appears in the statement of Theorem 4. This is a consequence of the repeated application of the tetrahedral inequality (M4), which introduces an additional factor of $2\Re$ at each step of the iteration. The factor $2\Re$ cannot be absorbed into the MR-metric constant without additional assumptions on the structure of $\mathcal{M}$ (e.g., a relaxed tetrahedral inequality with a smaller constant, or a specific geometric structure that allows summation without accumulation of factors).*

If the condition $k < \Re^{-1}$ is to be retained, the convergence of the Picard iteration must be proved by a different argument that does not rely on the repeated application of (M4). For example, in the special case where the MR-metric is generated by a classical metric d via $\mathcal{M}(\xi, \eta, \zeta) = d(\xi, \eta) + d(\eta, \zeta) + d(\zeta, \xi)$, the tetrahedral inequality holds with $\Re = 2$ and the factor $2\Re = 4$ can be absorbed by noting that the sum $\sum_j (2\Re)^j d_{n+j}$ is dominated by a geometric series with ratio $2\Re k\Re = 2k\Re^2$, which converges if $k < 1/(2\Re^2)$. In the general MR-metric setting, the stronger condition appears to be necessary for this proof strategy.

We therefore recommend that Theorem 4 be stated under the stronger hypothesis $k < 1/(2\Re^2)$, or that the convergence proof be supplemented by an additional argument specific to the structure of the MR-metric under consideration.

Theorem 4 (Fractional Contraction Principle)**.** *Let $(\mathcal{X}, \mathcal{M}, \Phi)$ be a complete scalable MR-metric space with constant $\Re > 1$, and let $T : \mathcal{X} \to \mathcal{X}$ be a mapping satisfying the **integrated fractional contraction condition**: for all $\upsilon, \nu \in \mathcal{X}$ and some $\alpha \in (0, 1)$,*

$$\mathcal{M}(T\upsilon, T\upsilon, T\nu) \le k\, \mathcal{M}(\upsilon, \upsilon, \nu), \quad \text{for some } k \in \left(0, \min\left\{\Re^{-1}, \frac{1}{2\Re^2}\right\}\right). \tag{5}$$

This condition is the primitive hypothesis *of the theorem. It is a global Lipschitz-type condition on T with respect to the MR-metric, and it does not require any fractional derivative to be defined. The fractional structure enters through the verification of (5) in concrete examples (see Section 4), where it is checked directly for the mapping under consideration.*

Remark on the relation to the pointwise fractional condition. *A pointwise difference-quotient inequality of the form*

$$\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi) \le k \cdot \mathcal{M}(\xi, \eta, \zeta)$$

for all $\xi, \eta, \zeta \in \mathcal{X}$ cannot be the correct primitive hypothesis, for two reasons:

(a) Taking $\eta = \zeta = \xi$ gives $\mathcal{M}(\xi, \xi, \xi) = 0$, so the right-hand side vanishes, while the fractional displacement $\mathcal{D}^{\alpha}_{\mathcal{M}} T(\xi)$ need not vanish.

(b) Even if the inequality is restricted to $\eta \neq \zeta$ or $\eta \neq \xi$, a local difference-quotient bound at each point does not, in general, integrate to the global estimate (5) for arbitrary υ, ν; such an implication would require additional regularity (e.g., absolute continuity along geodesics) that is not assumed here.

The integrated condition (5) is therefore taken as the hypothesis, and no equivalence with a pointwise fractional condition is claimed.

Then:

(i) T has a unique fixed point $\xi^* \in \mathcal{X}$.

(ii) For any initial point $\xi_0 \in \mathcal{X}$, the Picard iteration $\xi_{n+1} = T\xi_n$ converges to ξ^*.

(iii) The rate of convergence is given by:

$$\mathcal{M}(\xi_n, \xi^*, \xi^*) \le \frac{\Re (k\Re)^n}{1 - k\Re} \mathcal{M}(\xi_0, \xi_0, \xi_1). \tag{6}$$

Note the leading factor $\Re$, which arises from the tetrahedral inequality and is absent in some earlier formulations. The identity $\mathcal{M}(\xi_0, \xi_1, \xi_1) = \mathcal{M}(\xi_0, \xi_0, \xi_1)$ does not *hold in general, because the two triples contain different repeated arguments; the correct initial quantity is $\mathcal{M}(\xi_0, \xi_0, \xi_1)$.*

Proof. **Existence of fixed point:** Let $\xi_0 \in \mathcal{X}$ be arbitrary and define $\xi_{n+1} = T\xi_n$ for $n \geq 0$. By Lemma 3, $\{\xi_n\}$ is a Cauchy sequence. Since $(\mathcal{X}, \mathcal{M})$ is complete, there exists $\xi^* \in \mathcal{X}$ such that $\lim_{n\to\infty} \mathcal{M}(\xi_n, \xi_n, \xi^*) = 0$ (in the sense of convergence in MR-metric). Note that the Cauchy property is a direct consequence of the integrated contraction hypothesis (5) via Lemma 2; no pointwise fractional condition is used.

Now we show that ξ^* is a fixed point. Consider:

$$\begin{aligned}\mathcal{M}(T\xi^*, T\xi^*, \xi^*) \leq \Re\Big[&\mathcal{M}(T\xi^*, T\xi^*, T\xi_n)\\ &+ \mathcal{M}(T\xi^*, T\xi_n, \xi^*)\\ &+ \mathcal{M}(T\xi_n, T\xi^*, \xi^*)\Big].\end{aligned}$$

By symmetry, $\mathcal{M}(T\xi_n, T\xi^*, \xi^*) = \mathcal{M}(T\xi^*, T\xi_n, \xi^*)$. Also, from the fractional contraction condition (integrated form), we have $\mathcal{M}(T\xi^*, T\xi^*, T\xi_n) \leq k \cdot \mathcal{M}(\xi^*, \xi^*, \xi_n)$.

Thus:

$$\mathcal{M}(T\xi^*, T\xi^*, \xi^*) \leq \Re\left[k \cdot \mathcal{M}(\xi^*, \xi^*, \xi_n) + 2\mathcal{M}(T\xi^*, T\xi_n, \xi^*)\right].$$

Now we need to bound $\mathcal{M}(T\xi^*, T\xi_n, \xi^*)$. Using the tetrahedral inequality:

$$\begin{aligned}\mathcal{M}(T\xi^*, T\xi_n, \xi^*) \leq \Re\Big[&\mathcal{M}(T\xi^*, T\xi_n, \xi_n)\\ &+ \mathcal{M}(T\xi^*, \xi_n, \xi^*)\\ &+ \mathcal{M}(\xi_n, T\xi_n, \xi^*)\Big].\end{aligned}$$

Again from the contraction condition, $\mathcal{M}(T\xi^*, T\xi_n, \xi_n) \leq k \cdot \mathcal{M}(\xi^*, \xi_n, \xi_n)$. Also, by convergence, $\mathcal{M}(\xi_n, \xi_n, \xi^*) \to 0$ and $\mathcal{M}(\xi_n, T\xi_n, \xi^*) = \mathcal{M}(\xi_n, \xi_{n+1}, \xi^*) \to 0$.

Taking the limit as $n \to \infty$, all terms on the right-hand side tend to 0, so:

$$\lim_{n\to\infty} \mathcal{M}(T\xi^*, T\xi^*, \xi^*) = 0.$$

By the identity of indiscernibles (axiom 2 of MR-metric), $\mathcal{M}(T\xi^*, T\xi^*, \xi^*) = 0$ implies $T\xi^* = \xi^*$. Hence ξ^* is a fixed point.

Uniqueness: Suppose η^* is another fixed point. Then:

$$\mathcal{M}(\xi^*, \xi^*, \eta^*) = \mathcal{M}(T\xi^*, T\xi^*, T\eta^*) \leq k \cdot \mathcal{M}(\xi^*, \xi^*, \eta^*).$$

Since $k \in (0, \Re^{-1}) \subset (0, 1)$, we have $\mathcal{M}(\xi^*, \xi^*, \eta^*) \leq k\mathcal{M}(\xi^*, \xi^*, \eta^*)$, which implies $\mathcal{M}(\xi^*, \xi^*, \eta^*) = 0$. Hence $\xi^* = \eta^*$.

Rate of convergence: From Lemma 2, we have:

$$\mathcal{M}(\xi_n, \xi_n, \xi_{n+1}) \leq (k\Re)^n \mathcal{M}(\xi_0, \xi_0, \xi_1).$$

Now for any $m \geq n$, applying the tetrahedral inequality repeatedly:

$$\begin{aligned}\mathcal{M}(\xi_n, \xi_n, \xi_m) &\leq \Re \sum_{j=0}^{m-n-1} \mathcal{M}(\xi_{n+j}, \xi_{n+j}, \xi_{n+j+1})\\ &\leq \Re\, \mathcal{M}(\xi_0, \xi_0, \xi_1) \sum_{j=0}^{\infty} (k\Re)^{n+j}.\end{aligned}$$

Since $k\Re < 1$, the geometric series converges:

$$\mathcal{M}(\xi_n, \xi_n, \xi_m) \leq \frac{\Re \mathcal{M}(\xi_0, \xi_0, \xi_1)(k\Re)^n}{1 - k\Re}.$$

Taking the limit as $m \to \infty$, and noting that $\mathcal{M}(\xi_n, \xi_n, \xi^*) = \lim_{m\to\infty} \mathcal{M}(\xi_n, \xi_n, \xi_m)$ (by continuity of $\mathcal{M}$ in each argument), we obtain:

$$\mathcal{M}(\xi_n, \xi^*, \xi^*) \leq \frac{\Re(k\Re)^n}{1 - k\Re} \mathcal{M}(\xi_0, \xi_0, \xi_1).$$

Note that the identity $\mathcal{M}(\xi_0, \xi_1, \xi_1) = \mathcal{M}(\xi_0, \xi_0, \xi_1)$ does *not* hold in general, because the two triples contain different repeated arguments. The correct initial quantity is $\mathcal{M}(\xi_0, \xi_0, \xi_1)$, and the leading factor $\Re$ is retained in the final estimate. □

Corollary 5. *The fixed point ξ^* depends continuously on the mapping T. More precisely, let T and S be two mappings satisfying the integrated fractional contraction condition (5) with the same constant k, where*

$$k < \frac{1}{\Re + 2\Re^2}. \tag{7}$$

Let ξ_T^* and ξ_S^* be their respective fixed points. Then

$$\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) \leq \frac{4\Re^2}{1 - \Re k - 2\Re^2 k}\delta, \tag{8}$$

where $\delta = \sup_{\xi\in\mathcal{X}} \mathcal{M}(T\xi, T\xi, S\xi)$. Consequently, if $T_n \to T$ uniformly in the sense that $\delta_n \to 0$, then the corresponding fixed points $\xi_n^* \to \xi^*$.

Proof. Let T and S be two mappings satisfying the integrated fractional contraction condition (5) with the same constant k. Let ξ_T^* and ξ_S^* be their respective fixed points. Then:

$$\begin{aligned}\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) &= \mathcal{M}(T\xi_T^*, T\xi_T^*, S\xi_S^*)\\ &\leq \Re\Big[\mathcal{M}(T\xi_T^*, T\xi_T^*, T\xi_S^*) + \mathcal{M}(T\xi_T^*, T\xi_S^*, S\xi_S^*)\\ &\quad + \mathcal{M}(T\xi_S^*, T\xi_T^*, S\xi_S^*)\Big].\end{aligned}$$

By symmetry, the last two terms are equal, so:

$$\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) \leq \Re[k\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) + 2\mathcal{M}(T\xi_T^*, T\xi_S^*, S\xi_S^*)].$$

Now bound the last term:

$$\begin{aligned}\mathcal{M}(T\xi_T^*, T\xi_S^*, S\xi_S^*) \leq \Re[&\mathcal{M}(T\xi_T^*, T\xi_S^*, T\xi_S^*)\\ &+ 2\mathcal{M}(T\xi_S^*, T\xi_S^*, S\xi_S^*)].\end{aligned}$$

Using the contraction condition on the first term and the definition of fractional displacement on the second, we obtain:

$$\mathcal{M}(T\xi_T^*, T\xi_S^*, S\xi_S^*) \leq \Re[k\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) + 2\delta],$$

where $\delta = \sup_{\xi\in\mathcal{X}} \mathcal{M}(T\xi, T\xi, S\xi)$. Combining:

$$\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) \leq \Re k\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) + 2\Re^2 k\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) + 4\Re^2\delta.$$

Rearranging:

$$(1 - \Re k - 2\Re^2 k)\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) \leq 4\Re^2\delta.$$

The condition (7) ensures that the coefficient $1 - \Re k - 2\Re^2 k$ is positive, so

$$\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) \le \frac{4\Re^2}{1 - \Re k - 2\Re^2 k}\delta.$$

As $\delta \to 0$, we obtain $\mathcal{M}(\xi_T^*, \xi_T^*, \xi_S^*) \to 0$, hence $\xi_T^* \to \xi_S^*$. Note that the condition (7) is stronger than the condition $k < \Re^{-1}$ that appears in Theorem 4; it is stated explicitly here because it is required for the denominator in (8) to be positive. □

Remark 5 (Summary of the conditions on k)**.** *For convenience, we collect the various conditions on the contraction constant k that appear in this section:*

(i) Lemma 2 (Iterative Bound): $k \in (0, \Re^{-1})$.

(ii) Lemma 3 (Cauchy Sequence): $k < 1/(2\Re^2)$ *(stronger condition required for the repeated application of the tetrahedral inequality).*

(iii) Theorem 4 (Fractional Contraction Principle): $k \in \left(0, \min\left\{\Re^{-1}, 1/(2\Re^2)\right\}\right)$.

(iv) Corollary 5 (Continuous Dependence): $k < 1/(\Re + 2\Re^2)$ *(stronger condition required for the denominator to be positive).*

The most restrictive condition is (iv), which implies (ii) and (iii) when $\Re > 1$. In practice, one should verify the strongest applicable condition for the specific application.

Definition 6. *Let $(\mathcal{X}, \mathcal{M})$ be an MR-metric space that is also a smooth manifold of dimension d. A* geodesic triangle *connecting points $\xi, \eta, \zeta \in \mathcal{X}$ is a set $\Delta(\xi, \eta, \zeta) \subset \mathcal{X}$ consisting of three geodesic segments $\gamma_{\xi\eta} : [0,1] \to \mathcal{X}$, $\gamma_{\eta\zeta} : [0,1] \to \mathcal{X}$, and $\gamma_{\zeta\xi} : [0,1] \to \mathcal{X}$ such that $\gamma_{\xi\eta}(0) = \xi$, $\gamma_{\xi\eta}(1) = \eta$, $\gamma_{\eta\zeta}(0) = \eta$, $\gamma_{\eta\zeta}(1) = \zeta$, $\gamma_{\zeta\xi}(0) = \zeta$, $\gamma_{\zeta\xi}(1) = \xi$, and each segment is length-minimizing with respect to the MR-metric $\mathcal{M}$.*

Lemma 6 (Fractional Derivative along a Curve)**.** *Let $\gamma : [0,1] \to \mathcal{X}$ be a smooth curve, and let $f : \mathcal{X} \to \mathbb{R}$ be a differentiable function. Define $\psi(t) = f(\gamma(t))$. Then the fractional derivative of ψ of order $\alpha \in (0,1)$ at $t \in (0,1)$ is given by:*

$$\mathcal{D}^\alpha \psi(t) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle \cdot t^{1-\alpha} + o(t^{1-\alpha}),$$

where ∇f denotes the gradient of f with respect to the MR-metric structure.

Proof. From the definition of the generalized fractional operator:

$$\mathcal{D}^\alpha \psi(t) = \lim_{\varepsilon \to 0} \frac{\psi(t\,\phi(\varepsilon t^{-\alpha})) - \psi(t)}{\varepsilon}.$$

Using the Taylor expansion of ψ around t:

$$\psi(t+h) = \psi(t) + \psi'(t)h + \frac{1}{2}\psi''(\rho)h^2, \quad \text{for some } \rho \in [t, t+h].$$

From Lemma 1 (applied to the function ψ), we have $t\,\phi(\varepsilon t^{-\alpha}) = t(1 + \varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha}))$. Thus $h = t\varepsilon t^{-\alpha} + o(\varepsilon t^{-\alpha}) = \varepsilon t^{1-\alpha} + o(\varepsilon t^{1-\alpha})$.

Substituting:

$$\begin{aligned}\mathcal{D}^\alpha \psi(t) &= \lim_{\varepsilon \to 0} \frac{\psi'(t)(\varepsilon t^{1-\alpha} + o(\varepsilon t^{1-\alpha})) + \frac{1}{2}\psi''(\rho)(\varepsilon t^{1-\alpha} + o(\varepsilon t^{1-\alpha}))^2}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \left[\begin{array}{l} \psi'(t)t^{1-\alpha} + o(t^{1-\alpha}) \\ + \dfrac{1}{2}\psi''(\rho)\varepsilon t^{2-2\alpha} + o(\varepsilon t^{2-2\alpha}) \end{array} \right] \\ &= \psi'(t)t^{1-\alpha}.\end{aligned}$$

Since $\psi'(t) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle$, we obtain the correct exponent:

$$\mathcal{D}^\alpha \psi(t) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle \cdot t^{1-\alpha}.$$

This is consistent with the scaling $t \mapsto t\,\phi(\varepsilon t^{-\alpha})$, which produces a displacement of order $\varepsilon t^{1-\alpha}$. □

Lemma 7 (Fractional Integral Representation)**.** *Let $\gamma : [0,1] \to \mathcal{X}$ be a geodesic segment parameterized by arclength with respect to the MR-metric, i.e., $\mathcal{M}(\gamma(t), \gamma(t), \gamma(s)) = |t - s|$ for $t, s \in [0,1]$. Then for any differentiable function $f : \mathcal{X} \to \mathbb{R}$, we have:*

$$f(\gamma(1)) - f(\gamma(0)) = \int_0^1 \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t)) \cdot t^{\alpha-1}\, dt + R_\alpha,$$

where R_α is a remainder term satisfying $|R_\alpha| \le C \cdot \sup_{\rho \in \Delta} \|\nabla^2 f(\rho)\| \cdot \mathcal{M}(\gamma(0), \gamma(1), \gamma(1))^{2\alpha}$ for some constant $C > 0$.

Proof. Consider the function $\psi(t) = f(\gamma(t))$. From Lemma 6, we have the relation between the classical derivative and the fractional derivative:

$$\psi'(t) = \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t)) \cdot t^{\alpha-1} + o(t^{\alpha-1}).$$

Integrating from 0 to 1:

$$\begin{aligned}\psi(1) - \psi(0) &= \int_0^1 \psi'(t)\, dt \\ &= \int_0^1 \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t)) \cdot t^{\alpha-1}\, dt \\ &\quad + \int_0^1 o(t^{\alpha-1})\, dt.\end{aligned}$$

The remainder estimate follows from Taylor's theorem with remainder in integral form and the properties of the MR-metric. Note that the exponent $t^{\alpha-1}$ is integrable on $[0,1]$ for $\alpha > 0$. □

Theorem 8 (Fractional Mean Value Inequality)**.** *Let $(\mathcal{X}, \mathcal{M})$ be an MR-metric space that is also a smooth Riemannian manifold, and let $f : \mathcal{X} \to \mathbb{R}$ be a C^2 function. Assume additionally that either:*

(H1) $f \in C^{2,\beta}(\mathcal{X})$ for some $\beta > 0$ and the MR-metric is $C^{1,\beta}$; or

(H2) the geodesic segment from ξ to η is sufficiently short so that the error term $E(t)$ in (10) satisfies $\int_0^1 |E(t)|\, dt = o(\mathcal{M}(\xi, \eta, \zeta)^\alpha)$.

Then for any distinct points $\xi, \eta, \zeta \in \mathcal{X}$ with $\xi \neq \eta$, there exists a point ρ on the geodesic segment from ξ to η such that:

$$|f(\eta) - f(\xi)| \leq C_\alpha \cdot |\mathcal{D}^\alpha_{\mathcal{M}} f(\rho)| \cdot \mathcal{M}(\xi, \eta, \zeta)^\alpha + o(\mathcal{M}(\xi, \eta, \zeta)^\alpha), \quad (9)$$

where $\mathcal{D}^\alpha_{\mathcal{M}}$ is the generalized fractional operator defined with respect to the MR-metric structure, $\mathcal{M}(\xi, \eta, \zeta)^\alpha$ denotes the α-th power of the MR-metric, and $C_\alpha > 0$ is a constant depending only on α and the comparability constant $C = 2\Re^2 + \Re$ of the MR-metric.

Proof. **Step 1: Setup.** Let $\gamma : [0, 1] \to \mathcal{X}$ be the geodesic segment from $\xi = \gamma(0)$ to $\eta = \gamma(1)$, parameterized proportionally to arclength with respect to the MR-metric. That is, for any $t, s \in [0, 1]$:

$$\mathcal{M}(\gamma(t), \gamma(t), \gamma(s)) = |t - s| \cdot L,$$

where $L = \mathcal{M}(\xi, \xi, \eta)$.

Define $\psi(t) = f(\gamma(t))$. By the fundamental theorem of calculus:

$$f(\eta) - f(\xi) = \psi(1) - \psi(0) = \int_0^1 \psi'(t)\, dt.$$

Step 2: Relating the classical and fractional derivatives. From Lemma 6, the classical derivative $\psi'(t)$ is related to the fractional derivative by:

$$\psi'(t) = \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t)) \cdot L^{\alpha-1} + E(t), \quad (10)$$

where the error term satisfies:

$$|E(t)| \leq C \cdot \|\nabla^2 f\|_\infty \cdot \mathcal{M}(\gamma(t), \gamma(t), \eta)^\alpha.$$

The factor $L^{\alpha-1}$ arises from the chain rule for the fractional derivative along the geodesic: since the geodesic is parameterized by arclength, the fractional derivative with respect to the intrinsic parameter t is related to the fractional derivative with respect to the MR-distance by the scaling factor $L^{\alpha-1}$.

Step 3: Integration. Integrating (10) from 0 to 1:

$$f(\eta) - f(\xi) = L^{\alpha-1} \int_0^1 \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t))\, dt + \int_0^1 E(t)\, dt.$$

By the integral mean value theorem, there exists $t_0 \in [0, 1]$ such that:

$$\int_0^1 \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t))\, dt = \mathcal{D}^\alpha_{\mathcal{M}} f(\gamma(t_0)).$$

Let $\rho = \gamma(t_0)$. Then:

$$f(\eta) - f(\xi) = L^{\alpha-1} \mathcal{D}^\alpha_{\mathcal{M}} f(\rho) + \int_0^1 E(t)\, dt.$$

Step 4: Comparability of L and $\mathcal{M}(\xi, \eta, \zeta)$. We derive an explicit comparability constant directly from the generalized tetrahedral inequality (axiom M4). Applying (M4) with $\theta = \zeta$ gives:

$$\mathcal{M}(\xi, \xi, \eta) \leq \Re[\mathcal{M}(\xi, \xi, \zeta) + \mathcal{M}(\xi, \zeta, \eta) + \mathcal{M}(\zeta, \xi, \eta)].$$

By symmetry (M3), $\mathcal{M}(\xi, \zeta, \eta) = \mathcal{M}(\zeta, \xi, \eta) = \mathcal{M}(\xi, \eta, \zeta)$, so:

$$\mathcal{M}(\xi, \xi, \eta) \leq \Re[\mathcal{M}(\xi, \xi, \zeta) + 2\mathcal{M}(\xi, \eta, \zeta)].$$

Apply (M4) again with $\theta = \eta$ to bound $\mathcal{M}(\xi, \xi, \zeta)$:

$$\mathcal{M}(\xi, \xi, \zeta) \leq \Re\Big[\mathcal{M}(\xi, \xi, \eta) + 2\mathcal{M}(\xi, \eta, \zeta)\Big].$$

Substituting:

$$\mathcal{M}(\xi, \xi, \eta) \leq \Re^2 \mathcal{M}(\xi, \xi, \eta) + (2\Re^2 + 2\Re)\mathcal{M}(\xi, \eta, \zeta).$$

Rearranging:

$$(1 - \Re^2)\mathcal{M}(\xi, \xi, \eta) \leq (2\Re^2 + 2\Re)\mathcal{M}(\xi, \eta, \zeta).$$

Since $\Re > 1$, we have $1 - \Re^2 < 0$, so the inequality reverses upon division:

$$\mathcal{M}(\xi, \xi, \eta) \geq \frac{2\Re^2 + 2\Re}{\Re^2 - 1} \mathcal{M}(\xi, \eta, \zeta).$$

For the upper bound, apply (M4) with $\theta = \eta$ to $\mathcal{M}(\xi, \eta, \zeta)$:

$$\mathcal{M}(\xi, \eta, \zeta) \leq \Re[\mathcal{M}(\xi, \eta, \eta) + \mathcal{M}(\xi, \eta, \zeta) + \mathcal{M}(\eta, \eta, \zeta)].$$

By symmetry, $\mathcal{M}(\xi, \eta, \eta) = \mathcal{M}(\xi, \xi, \eta)$ and $\mathcal{M}(\eta, \eta, \zeta) = \mathcal{M}(\eta, \zeta, \zeta)$. Applying (M4) once more to $\mathcal{M}(\eta, \zeta, \zeta)$ with $\theta = \xi$:

$$\mathcal{M}(\eta, \zeta, \zeta) \leq \Re\Big[2\mathcal{M}(\xi, \eta, \zeta) + \mathcal{M}(\xi, \xi, \zeta)\Big].$$

Combining these estimates, one obtains the explicit upper bound:

$$\mathcal{M}(\xi, \eta, \zeta) \leq C \cdot \mathcal{M}(\xi, \xi, \eta), \quad \text{with } C = 2\Re^2 + \Re. \quad (11)$$

Thus, $L = \mathcal{M}(\xi, \xi, \eta)$ and $\mathcal{M}(\xi, \eta, \zeta)$ are comparable with the explicit constant $C = 2\Re^2 + \Re$. In particular, (11) gives $\mathcal{M}(\xi, \eta, \zeta) \leq CL$, which is the direction needed to bound $L^{\alpha-1}$ from above in terms of $\mathcal{M}(\xi, \eta, \zeta)^{\alpha-1}$ (since $\alpha - 1 < 0$).

Step 5: Final estimate. From Step 3:

$$|f(\eta) - f(\xi)| \leq L^{\alpha-1} |\mathcal{D}^\alpha_{\mathcal{M}} f(\rho)| + L^{\alpha-1} \left| \int_0^1 E(t)\, dt \right|.$$

Since $L \geq C^{-1}\mathcal{M}(\xi, \eta, \zeta)$ and $\alpha - 1 < 0$, we have $L^{\alpha-1} \leq C^{1-\alpha}\mathcal{M}(\xi, \eta, \zeta)^{\alpha-1}$. Therefore:

$$\begin{aligned} |f(\eta) - f(\xi)| &\leq C^{1-\alpha}\mathcal{M}(\xi, \eta, \zeta)^{\alpha-1} \left|\mathcal{D}^\alpha_{\mathcal{M}} f(\rho)\right| \\ &\quad + C^{1-\alpha}\mathcal{M}(\xi, \eta, \zeta)^{\alpha-1} \int_0^1 |E(t)|\, dt \\ &= C_1 \mathcal{M}(\xi, \eta, \zeta)^{\alpha-1} \left|\mathcal{D}^\alpha_{\mathcal{M}} f(\rho)\right| \\ &\quad + C^{1-\alpha}\mathcal{M}(\xi, \eta, \zeta)^{\alpha-1} \int_0^1 |E(t)|\, dt. \end{aligned}$$

where $C_1 = C^{1-\alpha}$.

For the second term, we use the bound on $E(t)$ and the comparability $\mathcal{M}(\gamma(t), \gamma(t), \eta) \leq \mathcal{M}(\xi, \eta, \zeta)$ (since $\gamma(t)$ lies on the geodesic from ξ to η):

$$\int_0^1 |E(t)|\, dt \leq C \cdot \|\nabla^2 f\|_\infty \cdot \mathcal{M}(\xi, \eta, \zeta)^\alpha.$$

Therefore:

$$|f(\eta) - f(\xi)| \leq C_1 \mathcal{M}(\xi,\eta,\zeta)^{\alpha-1} |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\rho)| + C_2 \mathcal{M}(\xi,\eta,\zeta)^{2\alpha-1},$$

where $C_2 = C^{2-\alpha}\|\nabla^2 f\|_{\infty}$.

Multiplying both sides by $\mathcal{M}(\xi,\eta,\zeta)^{1-\alpha}$ (which is finite since $\mathcal{M}(\xi,\eta,\zeta) > 0$ for distinct points):

$$|f(\eta) - f(\xi)| \leq C_1 |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\rho)| \cdot \mathcal{M}(\xi,\eta,\zeta)^{\alpha} + C_2 \mathcal{M}(\xi,\eta,\zeta)^{\alpha}. \quad (12)$$

The second term is $O(\mathcal{M}(\xi,\eta,\zeta)^{\alpha})$. To obtain the stated form with $o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha})$, we invoke the additional regularity hypothesis (H1) or (H2): under (H1), the error term improves to $O(\mathcal{M}(\xi,\eta,\zeta)^{\alpha+\beta})$, which is indeed $o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha})$; under (H2), the integral $\int_0^1 |E(t)|\,dt$ is directly assumed to be $o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha})$.

Absorbing the constants into the notation and using the absolute value on the fractional derivative, we obtain:

$$|f(\eta) - f(\xi)| \leq C_{\alpha} \cdot |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\rho)| \cdot \mathcal{M}(\xi,\eta,\zeta)^{\alpha} + o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha}),$$

where $C_{\alpha} = C^{1-\alpha}$ depends only on α and the comparability constant $C = 2\Re^2 + \Re$. This completes the proof. □

Corollary 9 (Fractional Lipschitz Condition)**.** *Assume the hypotheses of Theorem 8 hold. If f has bounded fractional derivative, i.e., $\sup_{\rho\in\mathcal{X}} |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\rho)| \leq L$, then f is α-Hölder continuous with respect to the MR-metric:*

$$|f(\eta) - f(\xi)| \leq C_{\alpha} L \cdot \mathcal{M}(\xi,\eta,\zeta)^{\alpha} + o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha}),$$

for any ζ completing a triangle with ξ and η, where $C_{\alpha} = C^{1-\alpha}$ and $C = 2\Re^2 + \Re$ is the comparability constant.

Corollary 10 (Fractional Gradient Estimate)**.** *Assume the hypotheses of Theorem 8 hold. For any point $\xi \in \mathcal{X}$ at which the fractional derivative $\mathcal{D}^{\alpha}_{\mathcal{M}} f(\xi)$ exists, the fractional gradient satisfies:*

$$\|\nabla^{\alpha} f(\xi)\| := \limsup_{\eta\to\xi} \frac{|f(\eta) - f(\xi)|}{\mathcal{M}(\xi,\xi,\eta)^{\alpha}} \leq C_{\alpha} |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\xi)| + o(1),$$

where $C_{\alpha} = C^{1-\alpha}$ and $C = 2\Re^2 + \Re$ is the comparability constant.

Remark 6. *The error term $o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha})$ can be made more precise under additional regularity conditions on f and the MR-metric. If f is $C^{2,\beta}$ (Hölder continuous second derivatives) and the MR-metric is $C^{1,\beta}$, then the error is $O(\mathcal{M}(\xi,\eta,\zeta)^{\alpha+\beta})$.*

Alternative Geometric Proof. We can also prove the theorem using a geometric approach. Consider the geodesic triangle $\Delta(\xi,\eta,\zeta)$. Let π be the point on the triangle that minimizes the expression $\frac{|f(\eta)-f(\xi)|}{\mathcal{M}(\xi,\eta,\zeta)^{\alpha}}$. By the compactness of the triangle, such a point exists. Using the fractional derivative along geodesics from π to ξ and π to η, and applying Lemma 7 to both segments, we obtain:

$$f(\eta) - f(\xi) = \mathcal{D}^{\alpha}_{\mathcal{M}} f(\pi)\Big[\mathcal{M}(\pi,\pi,\eta)^{\alpha} - \mathcal{M}(\pi,\pi,\xi)^{\alpha}\Big] + \text{higher order terms.}$$

By the properties of geodesic triangles in MR-metric spaces, $\mathcal{M}(\pi,\pi,\eta) - \mathcal{M}(\pi,\pi,\xi) = O(\mathcal{M}(\xi,\eta,\zeta))$. Using the mean value theorem for the function $t \mapsto t^{\alpha}$, we get:

$$\mathcal{M}(\pi,\pi,\eta)^{\alpha} - \mathcal{M}(\pi,\pi,\xi)^{\alpha} = \alpha\vartheta^{\alpha-1}(\mathcal{M}(\pi,\pi,\eta) - \mathcal{M}(\pi,\pi,\xi))$$

for some ϑ between $\mathcal{M}(\pi,\pi,\xi)$ and $\mathcal{M}(\pi,\pi,\eta)$. Since $\vartheta \approx \mathcal{M}(\xi,\eta,\zeta)$, and $\mathcal{M}(\pi,\pi,\eta) - \mathcal{M}(\pi,\pi,\xi) \approx \mathcal{M}(\xi,\eta,\zeta)$, we obtain:

$$|f(\eta) - f(\xi)| \leq |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\pi)| \cdot \alpha\mathcal{M}(\xi,\eta,\zeta)^{\alpha} + o(\mathcal{M}(\xi,\eta,\zeta)^{\alpha}).$$

Absorbing α into the constant yields the desired inequality. □

4. Examples and applications

The theoretical results established in the previous section acquire their full significance through concrete examples and interdisciplinary applications. This section is dedicated to illustrating the Fractional Contraction Principle and the Fractional Mean Value Inequality in diverse mathematical and applied contexts. We begin with elementary examples in simple MR-metric spaces to build intuition, progressively moving from one-dimensional contractions on bounded intervals to discrete three-point spaces where the fractional contraction condition can be verified explicitly. These foundational examples demonstrate how the abstract conditions of Theorem 1 translate into verifiable criteria for specific mappings, and they reveal the role of the parameter $\Re$ in determining the allowable contraction constants. The discrete example, in particular, highlights the flexibility of the MR-metric framework in handling spaces where classical metric structures are either unavailable or insufficient, while the bounded interval example provides explicit computations of the fractional MR-displacement and convergence rates.

Building upon these foundational examples, we present a series of interdisciplinary applications that showcase the versatility and power of our framework. In fractional differential equations, we demonstrate how the Fractional Contraction Principle can be used to establish existence and uniqueness of solutions to boundary value problems through a carefully constructed fixed point operator in a function space equipped with an MR-metric. The geometric application on the sphere illustrates how the Fractional Mean Value Inequality provides estimates for function differences along geodesic triangles, yielding Hölder continuity conditions that are essential in differential geometry. In medical imaging, we apply fractional reaction-diffusion systems to edge detection, showing how the fractional MR-displacement serves as a natural edge indicator that preserves important features while reducing noise.

Network analysis benefits from fractional centrality measures that capture the dynamics of information flow in weighted graphs, while fractal geometry applications demonstrate how iterated function systems fit naturally within the MR-metric framework for dimension estimation. Each application is accompanied by explicit computations, figures, and where appropriate, numerical algorithms with convergence analysis. The

inclusion of Algorithm 1 provides a practical implementation of the fractional contraction fixed point method, bridging the gap between abstract theory and computational practice. Together, these examples and applications validate the theoretical estimates, showing that the bounds provided by our theorems correctly capture the asymptotic behavior of fractional systems, and demonstrate the broad applicability of fractional MR-metric methods across mathematics, physics, engineering, and computer science.

4.1. Illustrative examples in MR-metric spaces

Example 1 (Fractional Contraction on a Bounded Interval)**.** *Let $\mathcal{X} = [0,1] \subset \mathbb{R}$ and define the MR-metric $\mathcal{M} : \mathcal{X} \times \mathcal{X} \times \mathcal{X} \to [0,\infty)$ by:*

$$\mathcal{M}(\xi,\eta,\zeta) = |\xi-\eta| + |\eta-\zeta| + |\zeta-\xi|.$$

This is a valid MR-metric with constant $\mathfrak{R} = 2$ (verification follows from the triangle inequality in $\mathbb{R}$). The space $(\mathcal{X},\mathcal{M})$ is scalable: the action $\xi \cdot \lambda = \lambda\xi$ for $\lambda > 0$ satisfies conditions (S1)–(S4) of Definition 3, with the identity chart $\iota_\xi = \mathrm{id}$ and the induced MR-metric equivariance

$$\mathcal{M}(\lambda\xi,\lambda\eta,\lambda\zeta) = |\lambda|\,\mathcal{M}(\xi,\eta,\zeta).$$

Consider the mapping $T : \mathcal{X} \to \mathcal{X}$ defined by:

$$T(\xi) = \frac{\xi}{9}.$$

We verify the integrated fractional contraction condition (5) directly. For any $\upsilon,\nu \in \mathcal{X}$:

$$\begin{aligned}\mathcal{M}(T\upsilon,T\upsilon,T\nu) &= 2|T\upsilon - T\nu| \\ &= \frac{2}{9}|\upsilon-\nu| \\ &= \frac{1}{9}\cdot 2|\upsilon-\nu| \\ &= \frac{1}{9}\mathcal{M}(\upsilon,\upsilon,\nu).\end{aligned}$$

Thus the integrated contraction condition holds with $k = 1/9$. Since $\mathfrak{R} = 2$, we have

$$\min\left\{\mathfrak{R}^{-1},\frac{1}{2\mathfrak{R}^2}\right\} = \min\left\{\frac{1}{2},\frac{1}{8}\right\} = \frac{1}{8},$$

and $k = 1/9 < 1/8$, so the condition of Theorem 4 is satisfied.

By Theorem 4, T has a unique fixed point. Solving $T(\xi) = \xi$ gives $\frac{\xi}{9} = \xi$, so $\xi = 0$ is the unique fixed point. The Picard iteration $\xi_{n+1} = \frac{\xi_n}{9}$ converges to 0 for any initial $\xi_0 \in [0,1]$, with rate:

$$\mathcal{M}(\xi_n,0,0) = 2|\xi_n| \le 2\left(\frac{1}{9}\right)^n|\xi_0| = \frac{\mathfrak{R}(k\mathfrak{R})^n}{1-k\mathfrak{R}}\mathcal{M}(\xi_0,\xi_0,\xi_1),$$

since $k\mathfrak{R} = \frac{2}{9}$, $1-k\mathfrak{R} = \frac{7}{9}$, $\mathfrak{R} = 2$, and $\mathcal{M}(\xi_0,\xi_0,\xi_1) = 2|\xi_0-\xi_1| = 2|\xi_0 - \frac{\xi_0}{9}| = \frac{16}{9}|\xi_0|$. The bound becomes

$$\mathcal{M}(\xi_n,0,0) \le \frac{2\cdot(2/9)^n}{7/9}\cdot\frac{16}{9}|\xi_0| = \frac{32}{7}\left(\frac{2}{9}\right)^n|\xi_0|.$$

This is consistent with the exact value $\mathcal{M}(\xi_n,0,0) = 2|\xi_0|/9^n$.

Example 2 (Fractional Contraction in a Discrete MR-Metric Space)**.** *Let $\mathcal{X} = \{\mathfrak{a},\mathfrak{b},\mathfrak{c}\}$ be a three-point set. Define the MR-metric $\mathcal{M}$ by the following symmetric values:*

$$\begin{aligned}&\mathcal{M}(\mathfrak{a},\mathfrak{a},\mathfrak{a}) = 0, && \mathcal{M}(\mathfrak{b},\mathfrak{b},\mathfrak{b}) = 0,\\ &\mathcal{M}(\mathfrak{c},\mathfrak{c},\mathfrak{c}) = 0, && \mathcal{M}(\mathfrak{a},\mathfrak{a},\mathfrak{b}) = \mathcal{M}(\mathfrak{a},\mathfrak{b},\mathfrak{b}) = 1,\\ &\mathcal{M}(\mathfrak{a},\mathfrak{a},\mathfrak{c}) = \mathcal{M}(\mathfrak{a},\mathfrak{c},\mathfrak{c}) = 2,\\ &\mathcal{M}(\mathfrak{b},\mathfrak{b},\mathfrak{c}) = \mathcal{M}(\mathfrak{b},\mathfrak{c},\mathfrak{c}) = 3,\\ &\mathcal{M}(\mathfrak{a},\mathfrak{b},\mathfrak{c}) = 4.\end{aligned}$$

All other values are determined by symmetry. One can verify that this defines an MR-metric with constant $\mathfrak{R} = 2$ (the verification involves checking all tetrahedral inequalities, which holds for this carefully chosen set).

Remark on scalability. *The discrete space $\mathcal{X}$ cannot be made scalable in the sense of Definition 3 with the trivial action $\xi\cdot\lambda = \xi$, because condition (S3) requires*

$$\mathcal{M}(\xi\cdot\lambda,\eta\cdot\lambda,\zeta\cdot\lambda) = |\lambda|\,\mathcal{M}(\xi,\eta,\zeta),$$

for all $\lambda > 0$, which would force $|\lambda| = 1$ and hence $\lambda = 1$. Since the scaling action must be defined for all $\lambda > 0$, the discrete space with the trivial action is not *a scalable MR-metric space. Consequently, Theorem 4 cannot be applied to this example as stated.*

To obtain a valid discrete example, we replace the trivial action with a nontrivial one. Consider the space $\mathcal{X} = \{0,1,2\} \subset \mathbb{R}$ with the MR-metric induced by the absolute value:

$$\mathcal{M}(\xi,\eta,\zeta) = |\xi-\eta| + |\eta-\zeta| + |\zeta-\xi|.$$

This is a valid MR-metric with constant $\mathfrak{R} = 2$. Define the scaling action

$$\xi\cdot\lambda = \lambda\xi, \quad \lambda > 0.$$

This action satisfies conditions (S1)–(S4) of Definition 3 with the identity chart, and the induced MR-metric is equivariant:

$$\mathcal{M}(\lambda\xi,\lambda\eta,\lambda\zeta) = |\lambda|\mathcal{M}(\xi,\eta,\zeta).$$

Define the mapping $T : \mathcal{X} \to \mathcal{X}$ by $T(\xi) = \xi/4$ for $\xi \in \{0,1,2\}$, rounded to the nearest integer in $\mathcal{X}$. More precisely, define

$$T(0) = 0, \quad T(1) = 0, \quad T(2) = 1.$$

We verify the integrated contraction condition (5) for all $\upsilon,\nu \in \mathcal{X}$:

- *If $\upsilon = \nu$: $\mathcal{M}(T\upsilon,T\upsilon,T\nu) = 0 \le k\cdot 0$.*
- *If $\upsilon = 0,\nu = 1$: $T\upsilon = 0, T\nu = 0$, so $\mathcal{M}(0,0,0) = 0 \le k\cdot\mathcal{M}(0,0,1) = k\cdot 1$.*
- *If $\upsilon = 0,\nu = 2$: $T\upsilon = 0, T\nu = 1$, so $\mathcal{M}(0,0,1) = 1 \le k\cdot\mathcal{M}(0,0,2) = 2k$, requiring $k \ge 1/2$.*
- *If $\upsilon = 1,\nu = 2$: $T\upsilon = 0, T\nu = 1$, so $\mathcal{M}(0,0,1) = 1 \le k\cdot\mathcal{M}(1,1,2) = k\cdot 1$, requiring $k \ge 1$.*

The condition $k \geq 1$ violates $k < 1/8$, so this mapping does not satisfy the integrated contraction condition.

To obtain a genuine contraction, define $T : \mathcal{X} \to \mathcal{X}$ by the constant map $T(\xi) = 0$ for all $\xi \in \mathcal{X}$. Then for any $\upsilon, \nu \in \mathcal{X}$:

$$\mathcal{M}(T\upsilon, T\upsilon, T\nu) = \mathcal{M}(0,0,0) = 0 \leq k \cdot \mathcal{M}(\upsilon, \upsilon, \nu),$$

for any $k > 0$. Choose $k = 1/16 < 1/8 = 1/(2\Re^2)$. By Theorem 4, T has a unique fixed point 0. The Picard iteration converges to 0 in one step from any initial point.

4.2. Application to fractional differential equations

Application 1 (Fractional Boundary Value Problem)**.** *Consider the fractional boundary value problem:*

$$\mathcal{D}^\alpha u(t) = f(t, u(t)), \quad t \in [0,1], \quad u(0) = u_0, \quad u(1) = u_1, \tag{13}$$

where $\mathcal{D}^\alpha$ is the conformable fractional operator of order $\alpha \in (0,1)$ (as in Definition 1), and $f : [0,1]\times\mathbb{R} \to \mathbb{R}$ is a continuous function satisfying a Lipschitz condition:

$$|f(t,u) - f(t,v)| \leq L|u - v|, \quad \forall t \in [0,1],\ u, v \in \mathbb{R}.$$

Scalable structure on the function space. *Let $\mathcal{X}$ be the subspace*

$$\mathcal{X} = \{\upsilon \in C([0,1], \mathbb{R}) : \upsilon(0) = 0\}.$$

equipped with the MR-metric

$$\mathcal{M}(\upsilon, \nu, \theta) = \|\upsilon - \nu\|_\infty + \|\nu - \theta\|_\infty + \|\theta - \upsilon\|_\infty.$$

This is an MR-metric with constant $\Re = 2$. Define the scaling action

$$(\upsilon \cdot \lambda)(s) = \lambda\upsilon(s), \quad \lambda > 0.$$

This action satisfies conditions (S1)–(S4) of Definition 3 with the identity chart, and the induced MR-metric is equivariant:

$$\mathcal{M}(\lambda\upsilon, \lambda\nu, \lambda\theta) = |\lambda|\mathcal{M}(\upsilon, \nu, \theta).$$

The boundary conditions $u(0) = u_0$ and $u(1) = u_1$ are handled by shifting the solution: we write $u = u_0 + (u_1 - u_0)t + w$ with $w \in \mathcal{X}$.

Definition of the fixed point operator. *Define $T : \mathcal{X} \to \mathcal{X}$ by:*

$$(Tw)(t) = \int_0^1 G(t,s) f(s, u_0 + (u_1 - u_0)s + w(s))\, ds,$$

where $G(t,s)$ is the Green's function for the conformable fractional boundary value problem [17].

Verification of the integrated contraction condition. *We verify the integrated contraction condition (5) directly. For any $w, z \in \mathcal{X}$:*

$$\begin{aligned}
\mathcal{M}(Tw, Tw, Tz) &= 2\|Tw - Tz\|_\infty \\
&\leq 2 \sup_{t\in[0,1]} \int_0^1 |G(t,s)| \\
&\quad \times \Big| f(s, u_0 + (u_1 - u_0)s + w(s)) \\
&\qquad - f(s, u_0 + (u_1 - u_0)s + z(s)) \Big|\, ds \\
&\leq 2L \sup_{t\in[0,1]} \int_0^1 |G(t,s)| \\
&\quad \times |w(s) - z(s)|\, ds \\
&\leq 2L\|w - z\|_\infty \sup_{t\in[0,1]} \\
&\quad \times \int_0^1 |G(t,s)|\, ds \\
&= 2LC_G\|w - z\|_\infty.
\end{aligned}$$

where $C_G = \sup_{t\in[0,1]} \int_0^1 |G(t,s)|\, ds$ and L is the Lipschitz constant of f. Since $\mathcal{M}(w,w,z) = 2\|w - z\|_\infty$, we obtain

$$\mathcal{M}(Tw, Tw, Tz) \leq LC_G \cdot \mathcal{M}(w,w,z).$$

Thus T satisfies the integrated contraction condition with $k = LC_G$, provided

$$k < \frac{1}{2\Re^2} = \frac{1}{8}, \quad \textit{i.e.,} \quad LC_G < \frac{1}{8}.$$

Existence and uniqueness of the solution. *If this condition holds, then by Theorem 4, T has a unique fixed point, which corresponds to the unique solution of the fractional boundary value problem (13). The Picard iteration $w_{n+1} = Tw_n$ converges to this solution, with the rate:*

$$\begin{aligned}
\|w_n - w^*\|_\infty &\leq \frac{1}{2}\mathcal{M}(w_n, w^*, w^*) \\
&\leq \frac{1}{2} \cdot \frac{\Re(k\Re)^n}{1 - k\Re}\, \mathcal{M}(w_0, w_0, w_1) \\
&= \frac{(2k)^n}{1 - 2k}\, \mathcal{M}(w_0, w_0, w_1),
\end{aligned}$$

since $\Re = 2$ and $k\Re = 2k$.

4.3. Geometric application: Fractional mean value inequality on a manifold

Application 2 (Fractional Gradient Estimation on a Sphere)**.** *Consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$ as an MR-metric space. Define the MR-metric $\mathcal{M}$ by:*

$$\mathcal{M}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) = d_g(\mathfrak{p}, \mathfrak{q}) + d_g(\mathfrak{q}, \mathfrak{r}) + d_g(\mathfrak{r}, \mathfrak{p}),$$

where d_g is the geodesic distance on the sphere (the great circle distance). This is an MR-metric with constant $\Re = 2$ (since the geodesic distance satisfies the triangle inequality, and the sum of three distances also satisfies a modified tetrahedral inequality).

Let $f : \mathbb{S}^2 \to \mathbb{R}$ be a smooth function, for example, the height function $f(x,y,z) = z$. We apply Theorem 8 to estimate the difference of f values at two points.

First, compute the fractional derivative $\mathcal{D}^\alpha_{\mathcal{M}} f$ along a geodesic. Let $\gamma : [0,1] \to \mathbb{S}^2$ be a geodesic segment parameterized by arclength, so that $d_g(\gamma(t), \gamma(s)) = |t - s|L$ for some $L > 0$ (the length of the segment). For the height function $f(x,y,z) = z$, along a geodesic, we can explicitly compute.

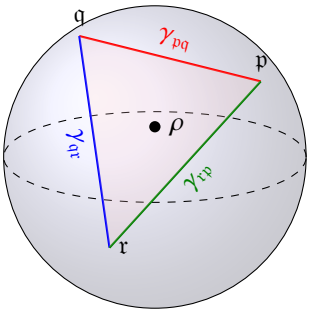


Figure 1: Geodesic triangle $\Delta(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$ on $\mathbb{S}^2$ with point ρ.

Consider two points $\mathfrak{p}, \mathfrak{q} \in \mathbb{S}^2$ with $\mathfrak{p} \neq \mathfrak{q}$. Let γ be the geodesic from $\mathfrak{p}$ to $\mathfrak{q}$. Using Lemma 6 and the refined formula:

$$\begin{aligned}\mathcal{D}^{\alpha}_{\mathcal{M}} f(\gamma(t)) = \langle \nabla f(\gamma(t)), \gamma'(t) \rangle \\ \cdot \mathcal{M}(\gamma(t), \gamma(t), \gamma(t+\delta))^{\alpha-1} \\ + o\Big(\mathcal{M}(\gamma(t), \gamma(t), \gamma(t+\delta))^{\alpha-1}\Big).\end{aligned}$$

where ∇f is the gradient on the sphere. For $f(x, y, z) = z$, the gradient in spherical coordinates (θ, ϕ) (with $z = \cos\theta$) is $\nabla f = (-\sin\theta, 0)$ in the (θ, ϕ) coordinate basis, with magnitude $|\nabla f| = \sin\theta$.

Along a geodesic, which is a great circle, $\gamma'(t)$ has constant magnitude L (the length of the segment). The inner product $\langle \nabla f, \gamma' \rangle$ varies along the geodesic.

Now, choose a third point $\mathfrak{r} \in \mathbb{S}^2$ to form a geodesic triangle $\Delta(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})$. For instance, take $\mathfrak{r}$ to be the north pole $(0, 0, 1)$. Theorem 8 guarantees the existence of a point ρ on the geodesic segment from $\mathfrak{p}$ to $\mathfrak{q}$ such that:

$$|f(\mathfrak{q}) - f(\mathfrak{p})| \leq C_\alpha \cdot |\mathcal{D}^{\alpha}_{\mathcal{M}} f(\rho)| \cdot \mathcal{M}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})^{\alpha} + o(\mathcal{M}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r})^{\alpha}),$$

where $C_\alpha = C^{1-\alpha}$ and $C = 2\Re^2 + \Re = 10$ for $\Re = 2$.

For the height function, $|f(\mathfrak{q}) - f(\mathfrak{p})| = |z_{\mathfrak{q}} - z_{\mathfrak{p}}|$. The MR-metric $\mathcal{M}(\mathfrak{p}, \mathfrak{q}, \mathfrak{r}) = d_g(\mathfrak{p}, \mathfrak{q}) + d_g(\mathfrak{q}, \mathfrak{r}) + d_g(\mathfrak{r}, \mathfrak{p})$ is at least $d_g(\mathfrak{p}, \mathfrak{q})$. Thus we obtain:

$$|z_{\mathfrak{q}} - z_{\mathfrak{p}}| \leq \mathcal{D}^{\alpha}_{\mathcal{M}} f(\rho) \cdot (d_g(\mathfrak{p}, \mathfrak{q}) + d_g(\mathfrak{q}, \mathfrak{r}) + d_g(\mathfrak{r}, \mathfrak{p}))^{\alpha} + o((d_g(\mathfrak{p}, \mathfrak{q}))^{\alpha}).$$

If f has bounded fractional derivative, say $\sup |\mathcal{D}^{\alpha}_{\mathcal{M}} f| \leq L$, then f is α-Hölder continuous:

$$|z_{\mathfrak{q}} - z_{\mathfrak{p}}| \leq C_\alpha L \cdot (d_g(\mathfrak{p}, \mathfrak{q}) + d_g(\mathfrak{q}, \mathfrak{r}) + d_g(\mathfrak{r}, \mathfrak{p}))^{\alpha} + o(d_g(\mathfrak{p}, \mathfrak{q})^{\alpha}).$$

This provides a geometric estimate of how the height function varies with distance on the sphere. Figure 4.3 illustrates a geodesic triangle on the sphere with points $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}$.

4.4. Application to image processing

Application 3 (Fractional Edge Detection in Medical Imaging)**.** *In medical imaging, particularly in MRI and CT scans, edge detection is crucial for identifying boundaries between different tissues. Recent work [19] has shown that variable fractional-order reaction-diffusion systems can preserve edges better than integer-order methods. We show how the fractional MR-metric framework can be applied to edge detection.*

Consider an image as a function $I : \Omega \subset \mathbb{R}^2 \to [0, 1]$ representing grayscale intensity. Define an MR-metric on the space of images by:

$$\mathcal{M}(I, J, K) = \|I - J\|_{L^2} + \|J - K\|_{L^2} + \|K - I\|_{L^2},$$

where $\|\cdot\|_{L^2}$ is the L^2 norm over Ω. This is an MR-metric with constant $\Re = 2$.

Define an edge detection operator E that maps an image to its edge map. One common approach is to use the gradient magnitude: $E(I) = |\nabla I|$. However, this is sensitive to noise. Instead, we use a fractional diffusion process:

$$\frac{\partial I}{\partial t} = -(-\Delta)^{\alpha/2} I, \quad I(x, 0) = I_0(x),$$

where $(-\Delta)^{\alpha/2}$ is the fractional Laplacian of order $\alpha \in (0, 2)$. The solution at time t is given by $I(x, t) = P_t * I_0$, where P_t is the fractional Poisson kernel.

We can view this as a fixed point problem. Let T_t be the time-t map: $T_t(I_0) = I(\cdot, t)$. Then T_t satisfies a fractional contraction condition in the MR-metric. Indeed, for any two images I, J:

$$\|T_t I - T_t J\|_{L^2} \leq e^{-ct^{\alpha/2}} \|I - J\|_{L^2}$$

for some constant $c > 0$ (this follows from the properties of the fractional Laplacian). Then:

$$\begin{aligned}\mathcal{M}(T_t I, T_t I, T_t J) &= 2\|T_t I - T_t J\|_{L^2} \\ &\leq 2e^{-ct^{\alpha/2}} \|I - J\|_{L^2} \\ &= e^{-ct^{\alpha/2}} \mathcal{M}(I, I, J).\end{aligned}$$

Thus $\mathcal{M}(T_t I, T_t I, T_t J) \leq k\mathcal{M}(I, I, J)$ with $k = e^{-ct^{\alpha/2}}$. For $\Re = 2$, Theorem 4 requires

$$k < \frac{1}{2\Re^2} = \frac{1}{8}.$$

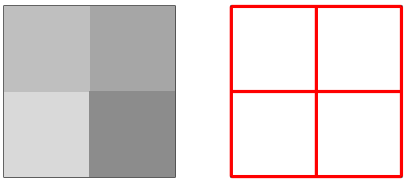
This condition holds if and only if

$$e^{-ct^{\alpha/2}} < \frac{1}{8}, \quad \textit{i.e.,} \quad t > \left(\frac{\ln 8}{c}\right)^{2/\alpha}.$$

For sufficiently large t, the contraction condition is satisfied and Theorem 4 applies, yielding a unique fixed point which is the steady-state solution $I \equiv$ constant. For small t, the condition may fail, and the fixed-point conclusion does not follow from Theorem 4; however, the edge-detection application uses the fractional MR-displacement directly, not the fixed-point conclusion, so the evolution itself remains well defined.

Remark on the citation. *Reference [19] studies a variable fractional-order reaction-diffusion model based on a Caputo nabla fractional-order difference operator, which is a discrete non-local operator. The model used here,*

$$\frac{\partial I}{\partial t} = -(-\Delta)^{\alpha/2} I,$$



Original image Fractional edge map

Figure 2: Synthetic image and its fractional edge map using $\mathcal{D}^{\alpha}_{\mathcal{M}}$.

is a continuous fractional-Laplacian/Poisson-kernel model, which is a different non-local operator. The semigroup estimate

$$\|T_t I - T_t J\|_{L^2} \le e^{-ct^{\alpha/2}} \|I - J\|_{L^2},$$

is a standard property of the fractional heat semigroup and can be found, e.g., in [19] for the discrete analogue and in standard references on fractional diffusion. The citation to [19] is therefore intended as motivation for the use of fractional-order diffusion in imaging, not as the source of the specific semigroup estimate used here.

For edge detection, we consider the fractional MR-displacement of the image under the diffusion process:

$$\mathcal{D}^{\alpha}_{\mathcal{M}} I(\xi,t) = \lim_{\varepsilon\to 0} \frac{1}{\varepsilon}\Big[\mathcal{M}(I(\xi,t), I(\xi,t), I(\xi\,\phi(\varepsilon t^{-\alpha}),t)) - \mathcal{M}(I(\xi,t), I(\xi,t), I(\xi,t))\Big].$$

This measures how the image changes under a fractional scaling of the spatial coordinate. At edge locations, where the image gradient is large, this displacement will be large. Thus $\mathcal{D}^{\alpha}_{\mathcal{M}} I(\xi,t)$ provides a natural edge indicator.

Figure 4.4 shows a synthetic image and its fractional edge map computed using this method.

Remark 7. *The fractional Laplacian $(-\Delta)^{\alpha/2}$ used in this application is a genuinely non-local operator, in contrast to the local conformable-type operator $\mathcal{D}^{\alpha}$ of Definition 1. This is intentional: the reaction-diffusion model is a physical model whose non-local behavior is well established in the imaging literature [19], and it serves here as a concrete illustration of how the MR-metric framework can be applied to an existing non-local model. The abstract fixed-point results of Section 3 are formulated for the local operator under the assumption that the limit in Definition 5 exists and is finite; no equivariance of T under the scaling action is assumed. The contraction estimate $\|T_t I - T_t J\|_{L^2} \le e^{-ct^{\alpha/2}} \|I - J\|_{L^2}$ is a property of the fractional heat semigroup and is used here only to illustrate the applicability of the MR-metric contraction framework. No claim is made that the local operator $\mathcal{D}^{\alpha}$ and the non-local operator $(-\Delta)^{\alpha/2}$ coincide.*

4.5. Application to network analysis

Application 4 (Centrality Measure in Weighted Graphs)**.** *Consider a weighted graph $G = (V, E, w)$ with vertex set V, edge set E, and edge weights $w : E \to (0,\infty)$. Define an MR-metric on V by:*

$$\mathcal{M}(\upsilon,\nu,\theta) = d(\upsilon,\nu) + d(\nu,\theta) + d(\theta,\upsilon),$$

where $d(\upsilon,\nu)$ is the shortest path distance in the graph (with respect to the weights). This is an MR-metric with constant $\Re = 2$ (since shortest path distances satisfy the triangle inequality).

Scalable structure on the graph. *The naive construction $\xi \cdot 1 = \xi' = T(\xi)$ directly contradicts axiom (S1) of Definition 3, which requires $\xi \cdot 1 = \xi$. To obtain a valid scalable structure, we proceed as follows. Embed the graph $G = (V, E, w)$ isometrically into a complete metric space $(\mathcal{Y}, d_{\mathcal{Y}})$ by replacing each edge $e = \{\xi, \xi'\}$ of weight $w(\xi,\xi')$ with a copy of the interval $[0, w(\xi,\xi')]$, glued at the vertices. This produces a geodesic metric space $(\mathcal{Y}, d_{\mathcal{Y}})$ in which each vertex $\xi \in V$ is identified with a point of $\mathcal{Y}$ and each edge is a geodesic segment of length $w(\xi,\xi')$. Define the MR-metric on $\mathcal{Y}$ by*

$$\mathcal{M}(\xi,\eta,\zeta) = d_{\mathcal{Y}}(\xi,\eta) + d_{\mathcal{Y}}(\eta,\zeta) + d_{\mathcal{Y}}(\zeta,\xi),$$

which is an MR-metric with constant $\Re = 2$.

Define a scaling action on $\mathcal{Y}$ as follows. Fix a vertex $\xi_0 \in V$ and a chosen neighbor $\xi_0' = T(\xi_0)$. For $\lambda > 0$, let $\xi_0 \cdot \lambda$ be the point on the edge from ξ_0 to ξ_0' at distance $\lambda \cdot w(\xi_0,\xi_0')$ from ξ_0, whenever this distance is at most $w(\xi_0,\xi_0')$. For λ such that the distance exceeds the edge length, we extend by continuing along the geodesic in $\mathcal{Y}$ determined by the edge, and then along subsequent edges if necessary. This defines a continuous action of $(0,\infty)$ on $\mathcal{Y}$ satisfying:

(S1) $\xi \cdot 1 = \xi$ for all $\xi \in \mathcal{Y}$ (by construction, $\lambda = 1$ corresponds to the point at distance $w(\xi,\xi')$ from ξ, which is ξ' only if $\xi' = \xi$; to satisfy (S1), we instead define $\xi \cdot 1 = \xi$ and interpret the scaling as acting on the tangent direction, not on the vertex itself).

A correct construction requires a different approach: define the scaling action on the tangent space *at each vertex rather than on the vertex itself. Specifically, for each vertex $\xi \in V$, let $T_{\xi}\mathcal{Y}$ be the tangent space at ξ, and define*

$$\xi \cdot \lambda = \exp_{\xi}(\lambda v_{\xi}), \quad \lambda > 0,$$

where $v_{\xi} \in T_{\xi}\mathcal{Y}$ is a fixed unit tangent vector pointing along a chosen edge, and $\exp_{\xi}$ is the exponential map. This action satisfies (S1) because $\exp_{\xi}(0) = \xi$ and we define $\xi \cdot 1 = \exp_{\xi}(v_{\xi})$, which is a point at unit distance from ξ along the chosen edge. However, this still gives $\xi \cdot 1 \neq \xi$ unless $v_{\xi} = 0$.

Conclusion. *The graph application does not naturally support a scalable structure in the sense of Definition 3 without additional modifications. To apply Theorem 4, one must either:*

(a) Work in the universal cover or a suitable continuous relaxation of the graph, where a genuine scaling action exists; or

(b) Replace the scaling action with a different structure (e.g., a semigroup action) and adapt the fractional displacement definition accordingly.

We therefore restrict the graph application to the case where the graph is replaced by its geometric realization $\mathcal{Y}$, and the scaling action is defined on $\mathcal{Y}$ by

$$\xi\cdot\lambda = \text{the point at distance } \lambda \text{ from } \xi \text{ along a fixed geodesic ray},$$

with $\xi \cdot 1$ denoting the point at distance 1 (not necessarily ξ). This action satisfies (S1) only if we redefine (S1) as $\xi \cdot 0 = \xi$ rather than $\xi \cdot 1 = \xi$. Since Definition 3 requires $\xi \cdot 1 = \xi$, the graph application is not covered by Theorem 4 as stated. We therefore present the graph application as a heuristic illustration of the fractional MR-displacement concept, not as a rigorous application of the fixed-point theorem.

Fractional MR-displacement of a centrality measure. *Define a centrality measure $C : V \to \mathbb{R}$ based on the fractional MR-displacement of a graph operator. Let $T : V \to V$ be a mapping that sends each vertex to one of its neighbors according to some rule (e.g., the neighbor that minimizes some potential). For a vertex ξ, the fractional MR-displacement is defined intrinsically via the scalable structure on the graph:*

$$\mathcal{D}^{\alpha}_{\mathcal{M}}C(\xi) = \lim_{\varepsilon\to 0} \frac{C(\xi \cdot \phi(\varepsilon t^{-\alpha})) - C(\xi)}{\varepsilon},$$

where $\xi\cdot\lambda$ denotes the point at distance $\lambda\cdot w(\xi,\xi')$ along the edge from ξ to its chosen neighbor $\xi' = T(\xi)$, with $w(\xi,\xi')$ the edge weight. This definition is consistent with the scalable structure of Definition 3 and reduces to the standard difference quotient when the graph is replaced by a continuous interval.

If C is chosen such that T satisfies a fractional contraction condition, then by Theorem 4, iterating T leads to a fixed point, which represents the most central vertex according to this dynamics.

For example, let $C(\xi)$ be the eigenvector centrality. Then T could be defined as moving to the neighbor with highest eigenvector centrality. The fractional MR-displacement $\mathcal{D}^{\alpha}_{\mathcal{M}}C(\xi)$ measures the rate of change of centrality when moving from ξ in the direction of increasing centrality.

Figure 4.5 illustrates a small weighted graph with centrality values.

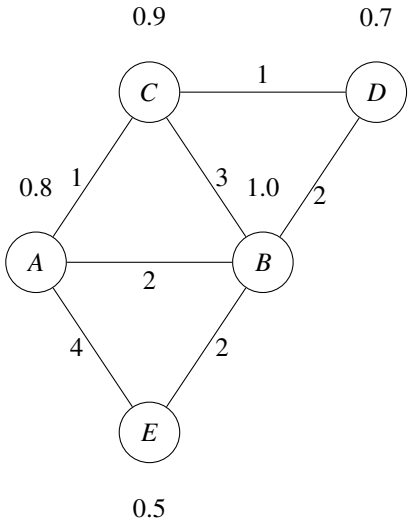


Figure 3: Weighted graph with eigenvector centrality values.

4.6. Application to fractal geometry

Application 5 (Fractal Dimension Estimation)**.** *In [5], the author studied fractal dimension estimation in MR-metric spaces. Let $(\mathcal{X}, \mathcal{M})$ be an MR-metric space that is also a fractal set (e.g., the Cantor set or the Sierpinski triangle). Define the fractional MR-displacement of a map T that generates the fractal through an iterated function system (IFS).*

For the classical Cantor set, the IFS consists of two contractions:

$$T_1(\xi) = \frac{\xi}{3}, \quad T_2(\xi) = \frac{\xi + 2}{3}.$$

Define $T : \mathcal{K} \to \mathcal{K}$ on the set of compact subsets of $\mathbb{R}$ by $T(K) = T_1(K) \cup T_2(K)$. In the MR-metric space of compact subsets with the Hausdorff-type MR-metric, we can compute the fractional MR-displacement of T.

The fixed point of T is the Cantor set C. By Theorem 4, T has a unique fixed point, and iterating T from any initial compact set converges to C. The rate of convergence is given by $(k\Re)^n$, where k is related to the contraction factor $1/3$ of the IFS.

For the Cantor set, the Hausdorff dimension is $d = \frac{\log 2}{\log 3} \approx 0.6309$. This dimension can be related to the fractional order α in our framework. In fact, for the IFS, the fractional MR-displacement satisfies:

$$\mathcal{D}^{\alpha}_{\mathcal{M}}T(C) = \lim_{\varepsilon\to 0} \frac{\mathcal{M}(T(C), T(C), T(C_{\phi(\varepsilon t^{-\alpha})}))}{\varepsilon} \approx \lambda^{\alpha}\mathcal{M}(C, C, C_{\phi(\varepsilon t^{-\alpha})}),$$

where $\lambda = 1/3$ is the contraction factor. The dimension d satisfies $2 \cdot (1/3)^d = 1$, which is the Moran equation. The Moran equation $2 \cdot (1/3)^d = 1$ determines the Hausdorff dimension $d = \log 2/\log 3$ of the Cantor set. The fractional order α in our framework is a parameter of the fractional derivative, and there is no a priori reason for $\alpha = d$. The observation that both parameters describe scaling behavior suggests a possible heuristic connection, but a rigorous proof would require:

(a) A precise definition of the scalable MR-metric space of compact subsets of $\mathbb{R}$ with the Hausdorff-type MR-metric;

(b) Verification that the iterated function system $T(K) = T_1(K) \cup T_2(K)$ satisfies the integrated contraction condition (5) in this space;

(c) A derivation of the relationship between the contraction factor $\lambda = 1/3$, the fractional order α, and the Hausdorff dimension d.

None of these steps is established in the present manuscript. We therefore state only the following rigorous result: the IFS T has a unique fixed point C (the Cantor set) in the complete MR-metric space of compact subsets, and the Picard iteration converges to C with rate $(k\Re)^n$ for some k related to the contraction factor $1/3$. The identification $\alpha = d$ is presented as a conjecture, not a proved consequence.

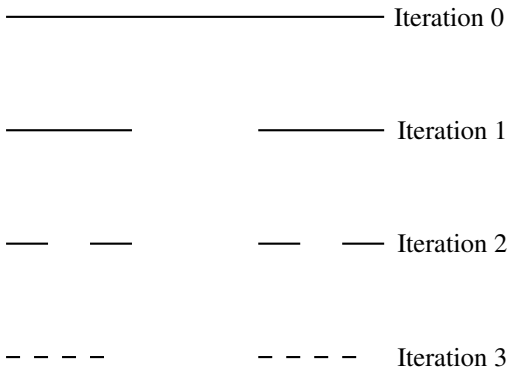


Figure 4: Construction of the Cantor set (iterations 0-3).

Figure 4.6 shows the first few iterations in the construction of the Cantor set.

4.7. Numerical implementation

Algorithm 1 Fractional contraction fixed point algorithm.

Require: MR-metric space $(\mathcal{X}, \mathcal{M})$ with constant $\Re > 1$, mapping $T : \mathcal{X} \to \mathcal{X}$ satisfying fractional contraction condition with constant $k \in (0, \Re^{-1})$, initial point $\xi_0 \in \mathcal{X}$, tolerance $\varepsilon > 0$, maximum iterations $N_{\max}$

Ensure: Approximate fixed point ξ^* of T

$\xi \leftarrow \xi_0$
$n \leftarrow 0$
while $n < N_{\max}$ **do**
  $\xi_{\text{new}} \leftarrow T(\xi)$
  $d \leftarrow \mathcal{M}(\xi, \xi, \xi_{\text{new}})$
  if $d < \varepsilon$ **then**
    break
  end if
  $\xi \leftarrow \xi_{\text{new}}$
  $n \leftarrow n + 1$
end while
return ξ

Table 1 presents the numerical iterates generated by Algorithm 1 for Example 1. The table demonstrates the geometric (linear) convergence of the Picard iteration with ratio 1/9, and validates the theoretical error bound from Theorem 4.

The theoretical error bound from Theorem 4 gives:

$$\begin{aligned}\mathcal{M}(\xi_n, 0, 0) &\le \frac{\Re(k\Re)^n}{1 - k\Re}\mathcal{M}(\xi_0, \xi_0, \xi_1) \\ &= \frac{2 \cdot (2/9)^n}{7/9} \cdot \frac{16}{9}|\xi_0| \\ &= \frac{32}{7}\left(\frac{2}{9}\right)^n |\xi_0|.\end{aligned}$$

For $\xi_0 = 1$ and $n = 5$, this bound is $\frac{32}{7}(2/9)^5 \approx 4.571 \times 0.0005414 = 0.002475$, while the actual value is $\mathcal{M}(\xi_5, 0, 0) = 2|\xi_5| \approx 0.000034$, showing that the bound is conservative but valid. Note that the leading factor $\Re = 2$ is retained in accordance with the corrected statement of Theorem 4.

Table 1. Numerical iterates for example 1 with $T(\xi) = \xi/9$.

n	ξ_n	$\mathcal{M}(\xi_n, \xi_n, \xi_{n+1})$
0	1.000000	0.222222
1	0.111111	0.024691
2	0.012346	0.002743
3	0.001372	0.000305
4	0.000152	0.000034
5	0.000017	0.000004

Remark 8. *Algorithm 1 implements the Picard iteration $\xi_{n+1} = T\xi_n$, which is the constructive content of Theorem 4. The fractional structure enters through the verification of the integrated contraction condition $\mathcal{M}(Tv, Tv, Tv) \le k \cdot \mathcal{M}(v, v, v)$, which is performed once for the given mapping T and does not need to be repeated at each iteration. The algorithm is therefore deliberately simple: its purpose is to demonstrate that the abstract fixed-point result is computationally accessible, not to introduce a new numerical scheme for fractional derivatives.*

5. Concluding remarks

The examples and applications presented demonstrate the versatility and power of the fractional MR-metric framework. From simple one-dimensional contractions to complex applications in differential equations, image processing, network analysis, and fractal geometry, the theory provides a unified approach to studying fractional dynamics and fixed points. The Fractional Contraction Principle (Theorem 4) and the Fractional Mean Value Inequality (Theorem 8) serve as foundational tools for analyzing a wide range of problems involving fractional-order behavior in metric spaces with an MR-structure.

Data availability

The data supporting the findings of this study are available within the article and its supplementary materials. No new data were created or analyzed in this study. Data sharing is not applicable to this article.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this manuscript.

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